Quality Growth: From Process to Product Innovation along the Path of Development*

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Abstract

We propose a demand-driven growth theory where process innovations and product innovations fulfil sequential roles along the growth path. Process innovations must initially set the economy on a positive growth path. However, process innovations alone cannot fuel growth forever, as their benefits display an inherent tendency to wane. Product innovations are therefore also needed for the economy to keep growing in the long run. When the economy fails to switch from a growth regime steered by process innovation to one driven by product innovation, R&D effort and growth will eventually come to a halt. However, when the switch to a product innovation growth regime does take place, a virtuous circle gets ignited. This happens because product innovation effort not only keeps growth alive when incentives to undertake process innovation diminish, but it also regenerates profit prospects from further process innovation effort.

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1 Introduction

Process and product innovations are two key determinants behind sustained economic growth. Process innovations introduce technological improvements that allow an expansion in the quantity of goods that an economy can deliver. Product innovations foster growth instead by bringing to the market goods of higher quality than those previously available. This paper presents a demand-driven growth theory in which both types of innovations fulfil crucial roles, and where their respective roles display a specific sequential timing. Our theory shows that process innovations must precede product innovations along the path of development. Yet, while process innovations can initially set the economy on a positive growth path, they cannot sustain rising incomes perpetually. Long-lasting growth requires that the economy is also able to start generating product innovations at some point. The reason for this is that, without the help of quality-upgrading innovations, the incentives to invest in process innovations will eventually start to wane as physical production increasingly expands.

The model features an economy with a vertically differentiated good, potentially available in an infinite number of (vertically ordered) quality levels. All the quality levels are produced with technologies that use labor as their sole input. Both labor productivity and the degree of vertical differentiation are endogenous to the model. Labor productivity increases as a result of process innovations. In particular, process innovations lower the costs of production (in terms of hours of labor), leading to an increase in the physical quantities that may be produced with a given amount of labor. Product innovations instead allow the introduction of better quality versions of the vertically differentiated good.

Innovations are the outcome of purposeful research and development effort. Hence, investment in process and in product innovations will be the endogenous response to the potential profit associated to each of them. The underlying force leading to their different timings along the growth path stems from our demand side. Individuals exhibit nonhomothetic preferences along the quality dimension. In particular, their willingness to pay for varieties of higher quality increases as their incomes rise. An implication of this is that product innovations tend to become increasingly profitable along the growth path, since product innovators can charge higher mark-ups when they face richer consumers. However, our nonhomothetic demand structure entails also a flip side. At early stages of development, the economy must rely on process innovations as the source of income growth. This is because the low willingness to pay for quality by consumers with low incomes stifles profit opportunities for product innovators.

Our theory then shows that at early stages of development process innovation must become the leading actor. Product innovation takes over a more prominent role instead in mature economies. Furthermore, such transition from process to product innovation effort proves essential for sustaining growth in our model. In a context where individuals display decreasing marginal utility on physical consumption, process innovations bring about two opposing dynamic forces. On the one hand, they drive the marginal utility of consumption down. Thus, the prospects of *future* profits from process innovation are endogenously dampened by *current* process innovation efforts. On the other hand, the higher quantity of consumption allowed by process innovations is exactly what spurs profit prospects from product innovation effort. The tension between these two countervailing forces means that an economy may or may not succeed in eventually switching from a growth path steered by process innovation to one steered by product innovation. When it fails to do so, growth will eventually come to a halt due to the negative effect of decreasing marginal utility on process innovation profits.

The switch to a product innovation growth regime can in turn ignite a virtuous circle with further process innovation down the road. The incapacity of cost-cutting innovations to spur growth perpetually lies in that a continuous expansion in quantity of production must struggle against the decreasing marginal utility of (physical) consumption that it simultaneously leads to. This struggle makes it increasingly hard to keep profit prospects from process innovation high enough to sustain it forever. Product innovations are able to relax this inherent tension. In particular, by raising the intrinsic quality of goods offered in the economy, product innovations make the decreasing marginal utility of (physical) consumption less pressing, and thereby regenerate profit prospects from further process innovation effort.

Our theory yields thus a model where long-run growth stems from a positive feedback loop between cost-cutting and quality-enhancing innovations. The endogenous growth literature has produced several models where growth results from the interplay of different types of innovation effort, such as general purpose technologies and sector-specific complementary inputs [e.g., Bresnahan and Trajtenberg (1995), Helpman and Trajtenberg (1998a, 1998b)], fundamental research/breakthroughs and secondary development [e.g., Jovanovic and Rob (1990), Aghion and Howitt (1996), Redding (2002)], invention and learning-by-doing [e.g., Young (1993), Stein (1997)]. All these models share a common trait: there exists one fundamental source of long-run growth that interacts with a transitional and bounded source of growth linked to the fundamental technology. Our model departs from the notion of fundamental and secondary sources of technical change. We look at the specific case of process and product innovations as *potentially* independent sources of technical change, but *ultimately* unable to sustain unbounded growth without one another.

The number of articles dealing separately with either process or product innovation effort in the endogenous growth literature is huge. Yet, it is hard to find models where both are explicitly involved together in steering the economy along the growth path, while at the same time playing a distinctive role as growth engines.[1] One prominent example is Foellmi, Wuergler and Zweimüller (2014), who build a growth model with non-homothetic preferences where firms must choose between product innovations to introduce new luxury goods to be consumed only by the rich, or process innovations that turn luxuries into mass consumption goods also available to the poor. Their model depicts situations where this type of product cycle arises as the optimal behavior by firms, and use it to explain how some new goods first introduced during the 20th century have later on become available as mass consumption goods (e.g., automobiles, refrigerators, etc.).[2] The main focus of our model is somewhat different, as it studies how the interplay between process and product innovation efforts can sustain a continuous increase in incomes in the long run, and how the preeminence of each type of innovation shifts along the growth path. In that sense, our model is mostly concerned with how an economy may keep growing *beyond* a mass consumption economy, in a context where rising incomes increasingly tilt consumer preferences towards quality expansion (and away from quantity expansion).[3]

Besides Foellmi *et al* (2014), a few other articles have mingled together vertical and horizontal innovations; see, e.g., Peretto (1998), Peretto and Connolly (2007), Sorger (2011), Chu, Cozzi and Galli (2012), Akcigit and Kerr (2018), Flach and Irlacher (2018). These articles, however, have all remained within standard homothetic frameworks, where trade-offs and interactions faced by innovators when choosing between process and product innovations are

[1] Models where growth is the result of the of new technologies that allow an increase in physical production (i.e., process innovations) can be found in: Shleifer (1986), Aghion and Howitt (1992), Jones (1995), Kortum (1997). Examples of models where growth is driven by the introduction of final goods of higher quality than before (i.e., product innovations) are: Segerstrom et al. (1990), Grossman and Helpman (1991a, 1991b), Stokey (1991), Segerstrom (1998). A third type of innovation, which is neglected by our model, is that one that leads to a horizontal expansion in the variety of goods, as in Judd (1985), Romer (1990), Grossman and Helpman (1991c, Ch. 3), Young (1993). We relegate to the concluding section a brief discussion on the possible effects of introducing variety-expanding innovations within the context of our model.

[2] Matsuyama (2002) also studies an endogenous growth model where goods initially affordable to the rich become gradually mass consumption goods affordable to all individuals. In his model, however, technological change is not the result of purposeful R&D effort, but it arises because of industry-specific learning-by-doing.

[3] Foellmi and Zweimüller (2006) also present a demand-driven endogenous growth model where individuals display non-homothetic preferences. Their model differs from ours substantially, with two key differences: i) in their model there is no quality differentiation (their nonhomotheticities are the result of hierarchical preferences with a horizontal continuum of goods); ii) their model features only cost-cutting innovations, which is combined with a setup cost that must be incurred to open new sectors/product lines.

disconnected from changes of consumer behavior along the growth path. As a result, these models remain silent about the needed transition from process to product innovation in order to keep richer consumers' demands continuously unsatiated.

A key aspect behind our demand-driven growth model is therefore the nonhomotheticity of preferences along the quality dimension (i.e., the notion that willingness to pay for quality upgrading rises with income). This is in fact a property of the preference structure that has been previously incorporated in several trade models [e.g., Flam and Helpman (1987), Murphy and Shleifer (1997), Fajgelbaum, Grossman and Helpman (2011, 2015), Jaimovich and Merella (2012, 2015)], and it has also been widely supported both by household-level data based on consumer surveys [e.g., Bils and Klenow (2001)] and bilateral trade flows data [e.g., Hallak (2006) and Choi *et al.* (2009)]. In our model, quality upgrading arises endogenously as the result of firms' effort to cater to consumers with rising incomes.[4] In addition to such quality-upgrading effort in response to rising incomes, our theory also predicts the need of a gradual shift from process innovation to product innovation along the growth path.

The need to transition from a growth regime heavily grounded on cost-reducing innovations towards one more strongly reliant on quality-enhancing innovations is a notion that in fact resonates well with current policy discussions in economies facing up to the so-called "middle income trap". The most paradigmatic example is probably China, and its response to it is encapsulated in the strategic plan dubbed 'Made in China 2025'. This is a broad long-run growth platform, but primarily aimed at bolstering the quality of production in the Chinese manufacturing sector, and explicitly emphasizing 'quality over quantity'. In fact, the deep concern regarding quality of Chinese products has been openly voiced by China's Premier Li Keqiang, stating that a key challenge ahead for China is '[to] redouble its efforts to upgrade from a manufacturer of quantity to one of quality'.[5] Quite remarkably, and acknowledging the relevance of nonhomotheticities along the quality dimension of consumption, one of the stated rationales behind the 'Made in China 2025' initiative is 'to be able to meet the higher demand for quality and high-end products by the growing middle-income Chinese population'.

The risks faced by the Chinese economy have also been recently raised more formally by Zilibotti (2017), warning that a shift to innovation is a necessary condition for keeping Chinese growth at a similar pace as in recent years. Wei, Xie and Zhang (2017) expand on this issue by highlighting that, given the rising labour costs faced by Chinese firms, China must start

[4] This is also a phenomenon that has received support by a growing strand of empirical papers; e.g. Verhoogen (2008), Brambilla *et al.* (2012), Manova and Zhang (2012), Bas and Strauss-Kahn (2015), Flach (2016).

[5] See reference in http://www.china.org.cn/business/2015-03/30/content_35192417.htm.

focusing on quality upgrading, so that firms will not need to depend on production of low-cost goods anymore. One key aspect of our theory is that growing incomes no only eventually call for product innovations to take place, but that they also help spur the incentives to introduce newer and better goods, so as to substitute lower-quality versions of them. Recent evidence from Chinese consumers with rising incomes in Beerli, Weiss, Zilibotti and Zweimüller (forthcoming) also grant support to this point, showing the case of progressive saturation of the market for low-quality transportation vehicles and its substitution by higher-raked ones.[6]

Besides the explicit acknowledgement in the policy arena of eventual growth-saturation in the absence of quality-improving innovations, recent studies relying on firm-level data also exhibit systematic support for this notion. For example, Argente, Lee and Moreira (2019), using 12-digit product barcode data in the US, find that variation of product quality explains nearly 90% of the time-series dynamics in sales of consumer products over firms' life cycle. On the other hand, they show that cutting prices via cost-cutting innovations play only a secondary and minor role. In particular, they conclude that demand factors driven by preferences for improved varieties of consumer goods are the main factor behind the life-cycle patterns of firms' growth in the US.[7] Also based on US barcode data, Hottman, Redding and Weinstein (2016) show that between 50-75% of variation in firms sales in the US is explained by differences in the overall quality/appeal of the goods they offer. Similar results are also obtained by Garcia-Macia, Hsieh and Klenow (2019), using in this case US census data on firms' labour force growth: they find that over 60% of US firms' growth (measured by their labour force) are the result of quality improvements of the set of goods they produce. Firm-level data therefore consistently points out that, in developed economies like the US, quality-upgrading innovations are the key factor for firms to be able to remain on a positive growth path.[8]

[6] The idea of output quality-enhancing as a crucial factor to keep growth alive when confronting consumers with rising incomes was acknowledged as well by post-war Japan. This led to the Japanese productive reorganization strategy known as 'The Quality Revolution'. The main idea behind this strategy was to undertake a long sequence of drastic improvements in quality, with the ultimate aim to expand Japanese exports to richer consumers in the US and Europe. By the early 80s, the Japanese strategy had proved so successful that American firms had embark themselves in their own "quality revolution", as a way to restore their domestic market shares which had been grossly hit by high-quality manufacturing imports from Japan (Kolb and Hoover, 2012).

[7] Related evidence based on barcode data is also presented by Argente, Lee and Moreira (2018) who, exploiting the variation of GDP per head over the period 2007-2013, show that the intensity of product innovation by US firms has been strongly procyclical.

[8] Although there exists an empirical literature that investigates firms' R&D investment differentiating between process and product innovation [e.g., Cohen and Klepper (1996), Huergo and Jaumandreu (2004), Parisi *et al.* (2006), Griffith *et al.* (2006), Petrin and Warzynski (2012), Harrison *et al.* (2014), Peters *et al.* (2017)], none of

The above empirical studies have managed to separate the contribution to firms' growth of cost-cutting innovations versus quality-improving innovations, concluding that in developed economies it is the latter that takes the lion's share. Yet, when considering growth at the aggregate level, it is hard to argue that process and product innovations do not interact with each other and also reinforce one another. In fact, the overall intensity of R&D investment is found to be systematically higher in richer economies than in poorer ones, and this is the case both for process and product innovations. The model we present aims at providing a rationale for why cost-cutting and quality-upgrading innovations are *both* crucial for sustaining long-run growth, and feedback on each other along the growth path. At the same time, the model showcases a growth path where quality-upgrading innovations are particularly needed in order to set economies on a growth path that avoids coming to halt in a sort of "medium-income trap". Lastly, the model features a growth path where product innovations tend to gradually gain prevalence as the main growth engine of mature economies.

The rest of the paper is organized as follows. Section 2 presents the setup and main assumptions of the model. Section 3 studies a simplified framework where the only source of technical change is process innovations, showing that growth is eventually bound to come to a halt. Section 4 introduces product innovations, and shows that they may help sustaining growth for longer. Section 5 describes the main features of the long-run dynamics of the model. Section 6 provides some concluding remarks. All relevant proofs are relegated to the Appendices.

2 Setup of the Model

Life evolves in discrete time over an infinite horizon $\mathcal{T} = \{1, 2, ..., \infty\}$. The economy is populated by a non-overlapping sequence of households. Each household is alive only for one period. In each period of time along $\mathcal{T}$ only one single household is alive. Once a household ceases to exist, it is immediately superseded by a newly born household.

All households consist of two members: W and R. Household members denoted by W are endowed with one unit of time, which they supply inelastically to firms as labor. Those denoted by R are also endowed with one unit of time, but they can choose either to enjoy it as leisure

these papers uses data collected from a large and diverse sample of countries. These studies either use firm-level data from one single country (like Cohen and Klepper (1996) for US, Huergo and Jaumandreu (2004) for Spain, Parisi *et al.* (2006) for Italy, Petrin and Warzynski (2012) for Denmark, or Peters *et al.* (2017) for Germany), or from a small number of countries with similar levels of income (like Griffith *et al.* (2006) and Harrison *et al.* (2014) for France, Germany, Spain and UK). As a result, this literature is silent about correlations between income per capita and intensity of investment in process innovation relative to product innovation.

time or, alternatively, to supply it as R&D effort to firms. We will often refer to W as *workers* and to R as *researchers*. Also, we will often index household members by the period $t \in \mathcal{T}$ during which they are alive; that is, W_t and R_t will denote, respectively, the worker and the researcher alive in t.

In addition to the households, the economy comprises also a non-overlapping sequence firms. Like households, firms are only active for one period of time, after which they are replaced by a new generation of firms. Setting up a firm entails no cost. Firms are risk neutral. Any profit made by a firm active in t is distributed to the household alive in t.

2.1 Technologies and Goods

The economy's output consists of a final consumption good produced by firms. The consumption good is *potentially* available in an unbounded number of vertically ordered quality levels: $q \in Q$. We normalize the lowest value of Q to unity. We will recurrently refer to the different quality levels as *quality versions*, and to the lowest quality ($q = 1$) as the *baseline quality*.

2.1.1 Prehistoric Technology

At the beginning of the time horizon ($t = 1$) the economy inherits a technology from the prehistoric period $t = 0$. The *prehistoric technology* allows transforming one unit of labor time into one unit of consumption good, but only in its baseline quality version $q = 1$. All firms active in $t = 1$ have free access to this technology.

In addition to having access to the prehistoric technology, firms may hire researchers to undertake R&D effort and create new technologies. Innovations can be of two different types: i) *Process innovations*, which raise labor productivity, but do not increase the quality of the consumption good; ii) *Product innovations*, which lead to the introduction of higher-quality versions of the consumption good, but leave labor productivity unaffected. We describe both types of innovations in greater detail in the next two subsections.

We assume that firms can innovate at most once during their lifetimes. When a firm generates an innovation, the firm is granted a patent on it. Once the firm ceases to exist, the patent expires and all the know-how contained in it becomes freely available to all other future generation of firms. We broadly refer to all the technologies that result from a defunct patent as *inherited technologies*. Inherited technologies may be further improved upon through *current* R&D effort (either in the form of process or product innovation effort).

2.1.2 New and Inherited Technologies I: the effects of process innovation

Consider a generic firm active in period t that hires the researcher R_t to undertake *process* innovation effort. Each unit of process innovation effort exerted by R_t allows the firm that hired the researcher to increase labor productivity by σ units above inherited technology's labor productivity.

In order to keep track of how technologies evolve over time, it proves convenient to define an indicator function $\mathbf{I}_p(t)$, which equals 1 when R_t is hired by a firm to undertake *process* innovation effort, and 0 otherwise. By using $\mathbf{I}_p(t)$, we can next define

$$A_{t-1} \equiv \sum\nolimits_{0}^{t-1} \mathbf{I}_p(\tau), \tag{1}$$

which denotes the (historical) number of researchers that have been hired by some firm to undertake process innovation effort before period t. Based on (1), we can now describe formally the set of technologies available to firms active in t resulting from process innovation effort.

Assumption 1 *Consider some generic period $t \in \mathcal{T}$, in which one of the active firms hires the researcher R_t to exert process innovation effort. The technologies available for producing the baseline quality variety (i.e., $q = 1$) in period t are given by:*

(i) Inherited technology: *Any firm active in t will be able to produce*

$$1 + \sigma A_{t-1}$$

units of $q = 1$ with one unit of labor, where $\sigma > 0$ and A_{t-1} is given by (1).

(ii) New technologies: *The firm that hired the researcher R_t will be able to produce*

$$1 + \sigma A_{t-1} + \sigma$$

units of $q = 1$ with one unit of labor, where $\sigma > 0$ and A_{t-1} is given by (1).

The first part of Assumption 1 stipulates that the effects of process innovation on labor productivity accumulate over time, and are (freely) transferred to *future* generation of firms as inherited technologies once the current innovator disappears. The second part describes the effects of *current* process innovation: a process innovation generated in period t increases labor productivity by $\sigma > 0$ units relative to the inherited technology received in t.

2.1.3 New and Inherited Technologies II: the effects of product innovation

Improved technologies may also originate from product innovation effort. Consider now a generic firm active in period t that hires R_t to undertake *product* innovation effort. Each unit of product innovation effort exerted by R_t allows the firm that hired the researcher to introduce an improved version of the consumption good that is $\rho > 0$ quality units above the quality version available before period t.

Analogously as done for process innovations, it proves convenient to define now an indicator function for the historical levels of product innovation effort, namely: $\mathbf{I}_q(t)$, which equals 1 when R_t is hired by a firm to undertake *product* innovation effort, and 0 otherwise. By means of $\mathbf{I}_q(t)$, we can next also define

$$q_t \equiv 1 + \rho \sum_0^t \mathbf{I}_q(\tau), \tag{2}$$

where q_t will be equal to highest quality version of the good available in period t.

Assumption 2 *Consider some generic period $t \in \mathcal{T}$, in which one of the active firms hires the researcher R_t to exert product innovation effort. Then:*

(i) Inherited technology: *Any firm active in t will be able to produce*

$$(1 + \sigma A_{t-1}) / q_{t-1} \tag{3}$$

units of the consumption good in the quality level q_{t-1}, where $q_{t-1} = 1$ if $t = 1$ and $q_t \equiv 1 + \rho \sum_0^{t-1} \mathbf{I}_q(t)$ for $t \geq 2$.

(ii) New technologies: *The innovative firm in t will be able to produce*

$$(1 + \sigma A_{t-1}) / q_t \tag{4}$$

units of the consumption good in the quality level q_t, where $q_t = q_{t-1} + \rho$.

Assumption 2 describes how a successful product innovation generated in period t allows the production of a version of the good whose quality level is $\rho > 0$ units above that one available before t. Two additional remarks about Assumption 2 are worth stressing here. Firstly, Assumption 1 and 2 taken together imply that past process innovations generate productivity improvements that are *not* quality-specific. More precisely, the numerators in (3) and (4) entail that improvements in labor productivity owing to *prior* process innovations apply identically to *all* the existing quality versions of the consumption good.[9] Secondly, the denominators in

[9] While the assumption that prior process innovations apply *identically* to all quality versions of the good may seem extreme, all our main results would still hold when process innovations spillovers applied only *partially* to higher-quality versions of the good, similarly to Jovanovic and Nyarko (1996) or Redding (2002).

(3) and (4) entail that the unit labor requirements are greater for higher quality versions of the consumption good.

2.2 Household Preferences

A household alive in period t is characterized by the following utility function:

$$U_t = \ln\left[\sum\nolimits_{q \in Q_t} [q\, x_t(q)]^q\right] + \eta(1 - \varepsilon_t). \tag{5}$$

In (5), $x_t(q) \in \mathbb{R}_+$ denotes the quantity of the quality version $q \in Q_t$ consumed by the household alive in period t, and $Q_t \subseteq Q$ is the set of quality versions available in t. The variable $\varepsilon_t \in \{0, 1\}$ takes the value of 1 if R_t decided to sell his time endowment as R&D effort to a firm (either in the form of process or product innovation), and 0 if he instead chose to use it as leisure, with $\eta > 0$ being the utility of leisure.[10]

Two important properties of (5) are worth stressing. Firstly, since the lowest value of the set Q_t is $q = 1$, the term $\sum_{q \in Q_t} [q\, x(q)]^q$ turns out to be a sum of convex functions in $x(q)$. As a consequence of this, in the optimum, households will select a *corner* solution for their consumption plan; that is, a solution characterized by $x(q) > 0$ for some $q \in Q_t$ and zero for all other quality versions. Secondly, the expression $[q\, x(q)]^q$ in (5) means that higher quality versions magnify the level of utility obtained from a given physical amount of the consumption good. Moreover, this magnifying effect becomes stronger the larger the value of physical consumption $x(q)$. This is a crucial feature of our model, as it will lead to a non-homothetic behavior in the demand for quality. In particular, the exponential effect of q on physical consumption $x(q)$ leads to demand functions where the willingness to pay for higher versions of the final good is increasing in the level of spending of the household.[11]

[10] The main role played by the utility of leisure parameter η is that of setting a shadow price for R&D effort. In addition, the fact that the leisure utility component enters additively into (5) implies that the R&D effort shadow price is independent of quantity of production and relative market prices. As it will become clear later on in Sections 3 and 4, a positive value of η opens up the possibility of R&D effort and growth coming to halt along the growth path. On the other hand, the increasing preeminence of product innovation along a positive long-run growth path is independent of the utility of leisure, but actually driven the nonhomotheticity of (5) along the quality dimension of consumption.

[11] In growth models with homothetic preferences, the distinction between reductions in costs per physical unit of production (i.e., process innovation) and increases in quality per physical unit (i.e., product innovation) may usually be blurred, since those sources of technical change can often be re-labeled to become isomorphic. Our nonhomothetic preference structure in (5) actually turns the distinction between quality improvements and physical productivity improvements economically meaningful, as the income-dependent willingness to pay for quality upgrading implies that those sources of growth cannot be taken simply as isomorphic to one another.

2.3 Equilibrium in Period t

In order to define the equilibrium of the economy in period t, it proves convenient to introduce some further auxiliary notation. We let $f_{k,t}$ denote a generic firm k that is active in t, and $\mathcal{F}_t$ denote the set of active firms in t. In addition, recalling that Q_t represents the set of quality versions available in period t, we let $Q_{k,t} \subseteq Q_t$ denote the subset of quality versions that firm $f_{k,t}$ is able to offer during t.

Definition 1 (Equilibrium in Period t) *An equilibrium in period t features: i) a vector $\left(\omega_{k,t}, r_{k,t}, \{p_{k,t}(q)\}_{q\in Q_{k,t}}\right)$ set by each active firm $f_{k,t} \in \mathcal{F}_t$, comprising a wage offer $\omega_{k,t} \in \mathbb{R}_+$, a researcher remuneration offer $r_{k,t} \in \mathbb{R}_+$, and collection of prices $p_{k,t}(q) \in \mathbb{R}_+$ for each $q \in Q_{k,t}$, and ii) a vector $\left(\{x_t(q)\}_{q\in Q_t}, \varepsilon_t\right)$ of household choices, comprising a collection of consumption choices $x_t(q) \in \mathbb{R}_+$ for each $q \in Q_t$, and an R&D effort level $\varepsilon_t \in \{0,1\}$ by R_t; such that:*

a) The worker W_t sells his time endowment to the firm $f_{k',t} \in \mathcal{F}_t$ whose $\omega_{k',t} = \max\{\omega_{k,t}\}_{f_{k,t}\in\mathcal{F}_t}$.

b) The researcher R_t sells his time endowment to the firm $f_{k'',t} \in \mathcal{F}_t$ whose $r_{k'',t} = \max\{r_{k,t}\}_{f_{k,t}\in\mathcal{F}_t}$, provided

$$U_t^*\left(\varepsilon_t = 1; \omega_{k',t}, r_{k'',t}, \left\{p_t^{\min}(q)\right\}_{q\in Q_t}\right) > U_t^*\left(\varepsilon_t = 0; \omega_{k',t}, r_{k'',t}, \left\{p_t^{\min}(q)\right\}_{q\in Q_t}\right), \tag{6}$$

while R_t sets $\varepsilon_t = 0$ when (6) fails to hold true.

c) The household alive in t maximizes (5), given the set of minimum prices $\left\{p_t^{\min}(q)\right\}_{q\in Q_t}$ at which each $q \in Q_t$ is offered in the market, and given their total earnings $\omega_{k',t} + \varepsilon_t \times r_{k'',t}$.

d) When (6) holds true, firm $f_{k'',t} \in \mathcal{F}_t$ requests R_t to exert the type of innovation effort (i.e., process or product) that yields the higher surplus to the firm.

e) No firm $f_{k,t} \in \mathcal{F}_t$ would be strictly better off if it offered a vector $\left(\widetilde{\omega}_{k,t}, \widetilde{r}_{k,t}, \{\widetilde{p}_{k,t}(q)\}_{q\in\widetilde{Q}_{k,t}}\right) \neq \left(\omega_{k,t}, r_{k,t}, \{p_{k,t}(q)\}_{q\in Q_{k,t}}\right)$, given: the technologies available to firm $f_{k,t}$, the choices made by all other firms $f_{j,t} \in \mathcal{F}_t$ with $j \neq k$, and the choices made by the household alive in t.

The equilibrium described above represents a situation in which the household in period t is a price taker in all markets, and chooses optimally their action given the constellation of prices set by firms in $\mathcal{F}_t$. That implies that they will sell their labor time endowment to one of the firms offering the highest wage, and they will only (possibly) buy a given quality version $q \in Q_t$ from a firm that charges the lowest price for that quality version. In addition, in case the household chooses to sell R_t's time endowment (rather than consuming it as leisure), they will sell it as well to one of the firms offering the highest researcher remuneration.

Although not explicitly stated in the equilibrium definition, given that there is free entry and no setup cost for firms, the equilibrium above will always lead to situations satifying zero-profit conditions by firms (otherwise, some firm in the set $\mathcal{F}_t$ could have a profitable deviation to exploit, contradicting condition '*e*' in the equilibrium definition).

One final aspect to notice from the equilibrium definition is that it postulates that firms are the agents that choose the type of R&D effort researchers must put, and that only one type of R&D effort can be exerted at a time.[12] In other words, households will only choose whether or not to sell R_t's time endowment to firms for a given remuneration, and it is firms that decide next how to use the researcher's time.[13]

3 Endogenous Growth via Process Innovation

In this section, we study the dynamic behavior of the economy under the assumption that only process innovations are feasible. Doing this allows a cleaner description of the conditions under which process innovation arises in equilibrium, and how it endogenously generates its own tendency to eventually come to a halt. In what follows, and throughout the whole paper, we take the baseline quality version of the consumption good as the *numeraire*.

Recall from Assumption 1 that all firms active in t can freely access a technology that allows them to produce $1+\sigma A_{t-1}$ units of baseline quality with one unit of labor. Competition among firms for worker W_t will thus lead to an equilibrium wage

$$\omega_t = 1 + \sigma A_{t-1}. \tag{7}$$

Firms will also compete for the R&D effort that the researcher R_t can supply to them. Competition by firms for R&D effort time implies that, in equilibrium, the researcher will be offered a total remuneration equal to the surplus it generates. Bearing in mind Assumption 1, it follows that the remuneration offered to R_t in equilibrium will have to be equal to σ per unit of effort time.[14]

[12] Appendix C briefly discusses how the model can be extended to allow firms to split R_t's time endowment between process innovation effort and product innovation effort.

[13] The equilibrium definition could, in principle, be extended to allow households decide *which* type of R&D effort to exert (rather than leaving that decision to firms), by letting firms offer different remuneration packages according to the type of R&D effort exerted by researchers.

[14] To see this, note that the firm that hired R_t in order to exert process innovation effort will be able to produce $1+\sigma A_{t-1}+\sigma$ units of the baseline quality good by hiring W_t, to whom it must pay $1+\sigma A_{t-1}$ as wage, leaving thus a surplus equal to σ to the firm.

Let now Y_t denote the lifetime earnings of the household alive at time t. Using (7), together with the fact that in equilibrium the researcher remuneration for process innovation effort will be equal to σ, $Y_t(\varepsilon_t)$ will thus given by:

$$Y_t(\varepsilon_t) = \begin{cases} 1 + \sigma A_{t-1} + \sigma, & \text{if } \varepsilon_t = 1, \\ 1 + \sigma A_{t-1} & \text{if } \varepsilon_t = 0. \end{cases} \tag{8}$$

From (8), the following result obtains quite straightforwardly.

Lemma 1 *Consider an economy in period t where the set of active firms $\mathcal{F}_t$ inherit a technology that allows them to produce $1+\sigma A_{t-1}$ units of the baseline quality with one unit of labor. Then, the equilibrium in period t will feature $\varepsilon_t^* = 1$ if and only if the following condition holds true:*

$$\ln\left(1 + \frac{\sigma}{1 + \sigma A_{t-1}}\right) > \eta. \tag{9}$$

The result in Lemma 1 states that an equilibrium where $\varepsilon_t^* = 1$ requires that the *additional utility* of consumption allowed by $Y_t(\varepsilon_t = 1)$, relative to that obtained from $Y_t(\varepsilon_t = 0)$, more than compensates the disutility of effort incurred when setting $\varepsilon_t = 1$.

Note from (9) that $\sigma/\left(1 + \sigma A_{t-1}\right)$ is strictly decreasing in A_{t-1}, and it converges to zero when $A_{t-1} \to \infty$. This implies that, for a sufficiently large value of A_{t-1}, condition (9) will fail to hold true, and period t will thus feature an equilibrium with $\varepsilon_t^* = 0$ (that is, an equilibrium with no process innovation effort). Furthermore, the ratio $\sigma/\left(1 + \sigma A_{t-1}\right)$ is strictly increasing in σ. As a result, the threshold value of A_{t-1} beyond which the equilibrium will fail to exhibit positive effort in process innovation tends to be greater for economies with a larger σ. The exact value of the threshold beyond which (9) fails to hold is pinned down by the level of A_{t-1} that equalizes the expression in the left-hand side of (9) to η. Namely,

$$\overline{A}(\sigma) \equiv \frac{1}{(e^{\eta} - 1)} - \frac{1}{\sigma}. \tag{10}$$

Given that we are interested in studying growth events where there is at least one instance with active R&D effort, we impose henceforth the following parametric restriction:

Assumption 3 $\sigma \geq e^{\eta} - 1$.

Notice that when Assumption 3 fails to hold, $\overline{A}(\sigma) < 0$, implying that the economy will *never* experience any process innovation in equilibrium, and the only technology available to firms throughout the entire time horizon $\mathcal{T}$ will thus be the *prehistoric technology.*

Combining (10) with the result in Lemma 1, we can observe that when Assumption 3 holds true, there is a maximum length of time during which the economy will be able to support an equilibrium with positive process innovation effort. The following proposition presents this argument formally.

Proposition 1 *Let Assumption 3 hold true, and let* $\bar{t} \equiv \text{integer}\{\overline{A}(\sigma)+1\}$*, where* $\overline{A}(\sigma)$ *is given by* (10)*. Then, the economy will experience an equilibrium with positive process innovation effort from* $t = 1$ *until* $t = \bar{t} \geq 1$*. In all* $t > \bar{t}$*, the economy will experience an equilibrium without process innovation effort. This, in turn, means that* $Y_t = 1+\sigma t$ *whenever* $t \leq \bar{t}$*, while* $Y_t = 1+\bar{t}$ *for all* $t > \bar{t}$*.*

Proposition 1 shows that, when $\sigma \geq e^{\eta} - 1$ (that is, when the increase in labor productivity generated by a process innovations is large enough relative to the disutility of R&D effort), the economy will keep investing in process innovation, as long as condition (9) keeps holding true. While $t \leq \bar{t}$, the economy will be on a positive growth path driven by rising labor productivity. This, in turn, translates into households' incomes that are growing over time according to $Y_t = 1 + \sigma t$. However, once the economy reaches a stage where $A_{t-1} > \overline{A}(\sigma)$, the equilibrium will cease to exhibit thereafter any further positive process innovation effort. From then on, growth will stop forever, and Y_t will remain constant at $Y_t = 1 + \bar{t}$.

The intuition behind the fact that process innovation eventually comes to halt hinges on the decreasing marginal utility of consumption. When the household alive in period t is contemplating whether or not to sell the time endowment of R_t as R&D effort to a firm, the household faces a trade-off between higher consumption versus higher leisure. Setting $\varepsilon_t = 1$ allows a level of consumption equal to $Y_t(\varepsilon_t = 1) = 1 + \sigma A_{t-1} + \sigma$. On the other hand, $\varepsilon_t = 0$ leads a *lower* level of consumption, $Y_t(\varepsilon_t = 0) = 1 + \sigma A_{t-1}$, but it yields an *additional* non-pecuniary benefit, η. As the economy goes through subsequent rounds of process innovation, both $Y_t(\varepsilon_t = 1)$ and $Y_t(\varepsilon_t = 0)$ increase. Yet, in a context with decreasing marginal utility of consumption, higher values of $Y_t(\varepsilon_t = 1)$ and $Y_t(\varepsilon_t = 0)$ imply that the consumption gap between them becomes gradually less appealing relative to the required leisure sacrifice η.

4 Introducing Product Innovations

We proceed now to introduce product innovation effort into the model. We consider the situation where the only quality version that was available before period t was the baseline quality. In this context, product innovation effort by R_t will allow introducing the quality version $q_t = 1+\rho$ to the market.

Before moving on to the full equilibrium analysis of the model, it proves convenient to first address the following two questions: *i*) what is the surplus that the product innovation yields to the firm that introduces it to the market?; *ii*) what is the price that the firm producing the quality version $q_t = 1 + \rho$ will be able to charge for this commodity?

Lemma 2 *Consider the researcher R_t alive in period t who is hired by a firm active in t to undertake product innovation effort. The surplus generated by the product innovation designed by R_t is given by:*

$$\pi_t^q = P_t \frac{1 + \sigma A_{t-1}}{1 + \rho} - (1 + \sigma A_{t-1}), \tag{11}$$

where P_t in (11) *denotes the market price of the quality version $q_t = 1 + \rho$, which in equilibrium will be given by:*

$$P_t = (1 + \rho) Y_t^{\rho/(1+\rho)}. \tag{12}$$

The first result in Lemma 2 shows the surplus (net of the labor cost) generated by a blueprint that may allow producing the higher-quality version ($q_t = 1 + \rho$), given its price P_t. Notice that competition by firms for the researcher's effort time entails that, in an equilibrium with positive product innovation effort, the remuneration offered to R_t will have to be equal to the RHS of (11). The second result, displayed by (12), shows that the price charged for the version of the good of quality $1+\rho$ rises with Y_t. The responsiveness of P_t to Y_t is a direct implication of our preference structure in (5), where the quality index q magnifies the utility derived from the physical quantity of consumption, $x(q)$. Such preference structure leads to a nonhomothetic behavior where the willingness to pay for the version of the good with $q = 1+\rho$ rises with Y_t.[15]

The equilibrium value of P_t thus depends on the level of Y_t. In the model, Y_t is itself also an equilibrium object. In particular, the level of Y_t will be ultimately a function of the innovation effort equilibrium choices in t. Equation (8) stated how Y_t rises with subsequent rounds of process innovation effort. The following lemma complements those results when we allow for product innovation effort as well.

Lemma 3 *Consider an economy that has carried out A_{t-1} rounds of process innovations effort before period t, and which has never carried out any product innovation effort before t. The set of active firms $\mathcal{F}_t$ will thus inherit a technology that will allow them to produce $1 + \sigma A_{t-1}$*

[15] The equilibrium value of P_t in (12) is that one that leaves the household alive in t indifferent between buying the higher-quality version at the price P_t and buying the baseline quality version (which being the *numeraire* carries a price equal to one). Given our nonhomothetic structure, this indifference price rises with Y_t.

units of the baseline quality good with one unit of labor. If an active firm in t hires R_t to exert product innovation effort, in equilibrium, the total income in t will be given by:

$$Y_t = (1 + \sigma A_{t-1})^{1+\rho}. \tag{13}$$

Lemma 3 shows that the larger the value of A_{t-1}, the greater Y_t will be in an equilibrium with product innovation effort. In addition, it is interesting to notice that when $A_{t-1} = 0$, the expression (13) yields $Y_t = 1$. In other words, in the absence of *any* previous rounds of process innovations, product innovation effort cannot induce by itself a rise in incomes. As a result, the model requires that some initial spells of process innovation effort take place in order to ignite income growth, when starting off from the level generated by the prehistoric technology. In turn, as consumers' incomes grow, this may endogenously generate a change in relative profits between the types of innovation effort, and shift at some point R&D effort from process innovation to product innovation. We next study how this particular growth sequence may arise as an equilibrium outcome of the model.

Before moving on, some additional notation is needed to distinguish the type of innovation effort carried out. We use henceforth $\varepsilon_{t,p} = 1$ for process innovation effort, and $\varepsilon_{t,q} = 1$ for product innovation effort. We continue to denote by $\varepsilon_t = 0$ the choice of consuming leisure.

4.1 Product Innovation Effort in a Generic t

We first briefly show that an equilibrium with product innovation cannot arise in $t = 1$. After that, we proceed to study whether an equilibrium with product innovation effort may eventually arise along the growth path at some generic point in time $t > 1$.

In an equilibrium where $\varepsilon^*_{1,q} = 1$ (that is, with positive product innovation effort at time $t = 1$) the income at $t = 1$ would follow from (13) with $A_0 = 0$. This would yield $Y_1 = 1$. Using Lemma 2, we can observe that this would in turn imply that $\pi^q_1 = 0$. As a consequence, no firm active in $t = 1$ would be willing to pay anything in exchange for product innovation effort by the researcher alive in $t = 1$, and hence no R&D effort would actually be offered to firms in the form of product innovation effort.

For an equilibrium with product innovation effort to arise in t two separate non-deviation conditions must be satisfied when $\varepsilon^*_{t,q} = 1$ holds. The first is that the household must not prefer to deviate to consuming R_t's time endowment as leisure. The second is that no firm active in t must find it profitable to request R_t to exert process innovation effort instead of product innovation effort. The previous paragraph shows that these non-deviation conditions fail to be satisfied at $t = 1$. The reason for this is that, in the absence of any previous process innovation effort (that is, when $A_{t-1} = 0$), equation (13) yields $Y_t = 1$, which (given our nonhomothetic preference structure) turns out to be too small to turn product innovation effort profitable. However, according to (13), the level of Y_t in an equilibrium with product innovation effort is increasing in A_{t-1}. This suggests that there *may* exist a value of A_{t-1} large enough to be able to support such an equilibrium. The next lemma lays out this result formally.

Lemma 4 *Consider an economy in a generic period t, such that $q_{t-1} = 1$, and which inherits a technology that allows firms in $\mathcal{F}_t$ to transform one unit of labor time into $1 + \sigma A_{t-1}$ units of the baseline quality variety. Suppose the researcher alive in t is offered a remuneration given by* (11)*, with P_t given by* (12)*, to exert product innovation effort for some firm $f_{k,t} \in \mathcal{F}_t$, and the worker alive in t is offered a wage given by* (7)*. Then:*

i) If

$$\rho \ln (1 + \sigma A_{t-1}) > \eta, \tag{14}$$

the household alive t will prefer selling the R&D effort time to $f_{k,t}$, instead of deviating to consuming R_t's time endowment as leisure.

ii) If

$$(1 + \sigma A_{t-1})^{1+\rho} > 1 + \sigma A_{t-1} + \sigma, \tag{15}$$

no single firm in $\mathcal{F}_t$ will find it profitable to request R_t to exert process innovation effort instead of product innovation effort.

Lemma 4 presents the two non-deviation conditions that will be able to support an equilibrium with product innovation in t. These conditions will only hold when A_{t-1} is sufficiently large. In other words, unless the economy has previously undergone a sufficiently large number of rounds of process innovations, it will *not* be able to sustain an equilibrium with product innovation effort in t. The next proposition shows that, for a sufficiently large A_{t-1}, an economy where the only quality variety available before t was the baseline one will indeed exhibit an equilibrium with positive product innovation effort at time t.

Proposition 2 *Consider an economy in period t, such that $q_{t-1} = 1$, and which has previously undergone A_{t-1} periods with positive process innovation effort. Then, the equilibrium in t will feature positive product innovation effort (i.e., in equilibrium, $\varepsilon^*_{t,q} = 1$) if and only if both* (14) *and* (15) *hold true.*

Proposition 2 shows that when the two non-deviation conditions stipulated in Lemma 4 are satisfied, the economy will exhibit an equilibrium with product innovation effort at time

t. In addition, this equilibrium is also unique. This implies that once an economy underwent enough rounds of process innovations in the past, so as the value of A_{t-1} that stems from (1) will be such that both (14) and (15) are satisfied, this economy will next switch at time t to an equilibrium where growth is pulled by product innovation effort. We next study whether such a switch will actually take place along the economy's growth path.

4.2 From Process Innovation to Product Innovation

Proposition 2 specifies the conditions that would lead to an equilibrium with positive product innovation effort in t. However, that proposition leaves one crucial question still unanswered: whether or not the economy's growth path will actually lead eventually to a value of A_{t-1} large enough to make (14) and (15) hold simultaneously. In fact, if it fails to do so (because the incentives to keep undertaking further process innovation wane too quickly), the equilibrium with product innovation effort as described by Proposition 2 will actually *never* materialize. We proceed to study now the conditions required for a successful transition from a equilibrium with process innovation to one with product innovation at some point along the growth path.

Some additional notation will prove useful for future reference. Firstly, we will denote by $\underline{A}_1$ the value of A_{t-1} that makes the LHS of (14) be equal to η. Namely:

$$\underline{A}_1 \equiv \frac{e^{\eta/\rho} - 1}{\sigma}. \tag{16}$$

Secondly, we will denote by $\underline{A}_2$ the value of A_{t-1} that solves $(1 + \sigma A_{t-1})^{1+\rho} = 1 + \sigma A_{t-1} + \sigma$. In this case, there is no general explicit solution for $\underline{A}_2$, which is thus defined implicitly by:

$$\frac{(1 + \sigma \underline{A}_2)^{1+\rho}}{1 + \sigma \underline{A}_2 + \sigma} \equiv 1. \tag{17}$$

Both $\underline{A}_1$ and $\underline{A}_2$ are strictly decreasing in σ.[16] When $A_{t-1} > \underline{A}_1$ and $A_{t-1} > \underline{A}_2$, the economy will feature an equilibrium with positive product innovation effort at time t. Since both conditions must hold for this, it is simpler to combine $\underline{A}_1$ and $\underline{A}_2$ together, and define:

$$\underline{A}(\sigma) \equiv \max\{\underline{A}_1, \underline{A}_2\}, \tag{18}$$

where in (18) we make the dependence of $\underline{A}$ on σ explicit. The threshold function $\underline{A}(\sigma)$ essentially pins down the value that A_{t-1} must necessarily reach for the economy to switch to an equilibrium with product innovation effort at time t.

[16] The fact that $\underline{A}_1$ decreases with σ can be observed directly from the expression in (16); a formal proof that $\underline{A}_2$ decreases with σ can be found in Appendix B.

By using the expressions in (16) and (17), we can pin down a cut-off level $\widehat{\sigma} > 0$ such that $\overline{A}(\sigma) \gtreqless \underline{A}(\sigma)$ when $\sigma \gtreqless \widehat{\sigma}$, where recall that $\overline{A}(\sigma)$ is given by (10).

Lemma 5 *Let*

$$\widehat{\sigma} \equiv e^{\eta/\rho} \left(e^{\eta} - 1\right). \tag{19}$$

When $\sigma = \widehat{\sigma}$, the threshold functions (10) *and* (18) *equal each other, that is: $\overline{A}(\widehat{\sigma}) = \underline{A}(\widehat{\sigma})$. Furthermore, $\overline{A}(\sigma) > \underline{A}(\sigma)$ whenever $\sigma > \widehat{\sigma}$, while $\overline{A}(\sigma) < \underline{A}(\sigma)$ whenever $\sigma < \widehat{\sigma}$.*

Considering the result in Proposition 1, an interesting implication of the result in Lemma 5 follows quite straightforwardly: a *sufficient* condition for an economy to be able to reach a value of A_{t-1} above the threshold $\underline{A}(\sigma)$ required for an equilibrium with product innovation effort to take place along its growth path is that σ is greater than $\widehat{\sigma}$. Notice from (19) that $\widehat{\sigma}$ is strictly decreasing in ρ and strictly increasing in η. Intuitively, a greater ρ makes the marginal surplus generated by product innovation effort larger, which in turn reduces the minimum level of labor productivity beyond which an economy will be able to switch to an equilibrium with product innovation. On the other hand, a greater η countervails the effect of ρ, by increasing the implicit shadow price of innovation effort, thereby discouraging R&D effort.

The following proposition describes more formally the conditions under which a successful transition to a growth-regime with product innovation effort will materialize along the economy's growth path.

Proposition 3 *Let Assumption 3 hold true. Then, there exists a threshold level $\widetilde{\sigma}$, where $e^{\eta/\rho}\left(1 - e^{-\eta}\right) < \widetilde{\sigma} < \widehat{\sigma} \equiv e^{\eta/\rho}\left(e^{\eta} - 1\right)$, such that:*

i) If $\sigma < \widetilde{\sigma}$, the economy will experience process innovation effort from $t = 1$ until some period $t = \bar{t} \geq 1$, where $\bar{t} \equiv$ integer$\{\overline{A}(\sigma) + 1\}$ with $\overline{A}(\sigma)$ given by (10)*, and it will stop carrying out any type of innovation effort for all $t > \bar{t}$.*

ii) If $\sigma \geq \widetilde{\sigma}$, the economy will experience process innovation effort from $t = 1$ until some period $t = \underline{t} \geq 1$, where $\underline{t} \equiv$ integer$\{\underline{A}(\sigma) + 1\}$ with $\underline{A}(\sigma)$ given by (18)*, and it will switch to an equilibrium with product innovation effort in $t = \underline{t} + 1$.*

Proposition 3 shows that when σ is sufficiently low, the economy will *never* manage to switch to an equilibrium with product innovation. In these cases, process innovation and income growth will eventually come to a halt a $t = \bar{t}$. From then on, the value of A_{t-1} will remain constant at a level equal to $A_{t-1} = \bar{t}$ for all $t > \bar{t}$. Incomes will, accordingly, also stay fixed thereafter at the level $Y_t = 1 + \sigma\bar{t}$.

On the other hand, for levels of σ that are large enough, an equilibrium with product innovation effort will actually arise at time $t = \underline{t}+1$. In such cases, the economy will experience an initial phase of growth driven by process innovation effort from $t = 1$ until $t = \underline{t}$. As incomes rise during this phase, the (implicit) willingness to pay for the higher-quality version increases. Eventually, at $t = \underline{t}$, Y_t will have risen enough so as to increase the willingness to pay for $q = 1 + \rho$ sufficiently to turn product innovation effort more profitable than process innovation effort in the following period. At this point, the growth-regime switch takes place at $t = \underline{t} + 1$.

It is interesting to highlight the impact of the different technological parameters in promoting a successful growth-regime switch along the growth path. Note from the result in Proposition 3 that a *sufficient* condition for this to take place is that $\sigma \geq \widehat{\sigma} \equiv e^{\eta/\rho} (e^{\eta} - 1)$.[17] This condition essentially shows that both a greater ρ and a greater σ are instrumental for this growth-regime switch to materialize. The reason for the former is indeed quite straightforward: a greater ρ increases the marginal productivity of R&D effort in product innovation. The reason why a greater σ is also helpful for the growth-regime transition is, however, quite different and much more subtle. In this case it is to do with the effect of nonhomothetic preferences on the willingness to pay for quality upgrading. More precisely, since a greater σ implies that the income effect generated by each round of process innovation effort is larger, in a context with nonhomotheticicities along the quality dimension of consumption, this helps speeding up the transition to a regime with quality-upgrading innovations.

5 Quality Upgrading and Long-Run Growth

In our model, process innovation effort exhibits an inherent tendency to come to a halt. For this reason, as Section 4.2 shows, it becomes crucial that the switch to an equilibrium with product innovation effort takes place soon enough (or, otherwise, the growth-regime switch will simply end up not happening at all). Naturally, product innovation effort helps sustaining positive growth while it takes place. However, there is one additional positive effect that product innovation exerts on growth: it may also boost the incentives to further undertake process innovation effort in future periods.

Growth driven by process innovations is constantly at risk of being thwarted by its ensuing

[17] Similarly, a sufficient condition for an economy not to be able to switch to an equilibrium with product innovation along the growth path is that $\sigma \leq e^{\eta/\rho} (1 - e^{-\eta})$. As a result, economies whose σ is too small, or whose ρ is too small, will tend to experience growth driven by process innovation during for some periods, after which growth will eventually come to a halt.

effect of the decreasing marginal utility of consumption. Product innovation effort works on a rather distinct dimension that does not face such a predicament: it spurs growth by allowing higher utility by each unit of physical consumption. Furthermore, the marginal utility of consumption declines more slowly for higher quality versions than for lower quality ones. Quality upgrading thus relaxes the depressing effect that decreasing marginal utility of consumption imposes on the incentives to further raise physical production via process innovation. As a result, an economy that manages to switch at some point along its growth path to an equilibrium with product innovation effort, will also see the incentives to exert further R&D effort become reinvigorated by product quality upgrading. The following lemma shows an important preliminary result regarding the long-run growth path of an economy that manages to transition from a process to a product innovation regime.

Lemma 6 *Consider an economy that satisfies $\sigma \geq \widetilde{\sigma}$, where $\widetilde{\sigma}$ is the threshold defined in Proposition 3 beyond which the growth-regime switch will take place. This economy will be able to sustain an equilibrium with some type of innovation effort during the entire infinite time horizon $\mathcal{T}$.*

Lemma 6 essentially states that economies which are able to switch to an equilibrium with product innovation at time $t = \underline{t} + 1$, will also be able sustain an equilibrium with *some* type positive innovation effort at any time $t' > \underline{t} + 1$. We proceed to study now *which* type of innovation effort actually takes place along the economy's growth path.

Proposition 4 *An economy will be able to sustain an equilibrium growth path with positive growth in the long run if and only if it satisfies $\sigma \geq \widetilde{\sigma}$. Along such a growth path, the economy experiences the following growth sequence:*

1. *There is an initial growth phase driven by process innovation effort starting in $t = 1$ until $\underline{t} \geq 1$, where $\underline{t} \equiv \text{integer}\{\underline{A}(\sigma) + 1\}$ with $\underline{A}(\sigma)$ given by (18).*

2. *At $t = \underline{t} + 1$ the economy switches to an equilibrium with product innovation effort. From $t = \underline{t} + 1$ onwards, the economy's growth path exhibits finite spells of growth driven by process innovation effort, alternating with finite spells of growth driven by product innovation effort.*

Proposition 4 shows that economies that manage to sustain positive growth in the long run will exhibit finite spells of growth driven by process innovation effort, alternating with finite spells of growth driven product innovation. Those sequences of finite spells alternate each other indefinitely. This result showcases the interplay between process and product innovations present in our model. On one side, the quantity expansion brought about by process innovations bolsters the incentives to start investing in quality-upgrading innovations. On the other side, the ensuing quality expansion stemming from product innovations relaxes the inherent tendency of profit prospects from further process innovations to decay. The alternation of equilibria with process and product innovation efforts exploits this feedback loop, and is thus instrumental to keeping income growth alive in the long run.

An interesting implication of Proposition 4 is that the growth path of a successful economy will display deterministic growth cycles. In particular, spells where growth rates start to wane owing to the eroding effect of quantity expansion on marginal utility of consumption alternate with periods of higher growth rates in which quality upgrading takes place. In turn, higher growth rates resulting from quality upgrading will reinvigorate growth rates from further quantity expansion. This dynamic mechanism somehow resembles the one featured in Matsuyama (1999), which highlights the alternation of growth cycles driven by capital accumulation (to expand the stock of capital) and innovation activities (to expand the variety of capital goods). In Matsuyama (1999), the decreasing marginal productivity of a fixed variety of capital goods implies that expanding the variety of capital goods is necessary for sustaining long-run growth. In our model, cycles emerge from a different source of interaction: the complementarities of independently meaningful types of innovation effort, whose relative prominence shifts endogenously along the growth path.[18] Notice that an important difference between the two models is to do with the origin of the underlying tension leading to growth cycles in each of them. In Matsuyama (1999), cycles are the result of supply-side features of the economy. By contrast, the cycles described by Proposition 4 are crucially linked to the non-homothetic demand structure of the model. In particular, it is owing to such non-homothetic structure that quality upgrades from product innovations keep reviving the demand side of the economy, and thus regenerating the incentives to carry out process innovations.[19]

[18] Shleifer (1986) and Francois and Lloyd-Ellis (2003) also present models that feature deterministic growth cycles. In those models cycles are the result of the endogenous clustering of innovations in certain moments of time due to the presence of (positive) demand externalities.

[19] Proposition 4 describes a growth path where the economy is moving back-and-forth between growth spells driven by process innovations and growth spells pulled by product innovations. However, the main intention of that result is not that of presenting those periodic discrete jumps as a truthful representation of the growth paths followed by developed economies, but to showcase the nature of the feedbacks between both types of innovation effort along the growth path as cleanly as possible.

5.1 Evolution of Process and Product Innovation along the Growth Path

Proposition 4 shows that an economy featuring long-run growth undergoes an initial growth phase propelled by process innovation effort, followed by a second phase where growth is driven by product innovation effort. After this initial sequence, the economy alternates indefinitely between spells of growth pulled by process innovations and spells of growth pulled by product innovations. The proposition then leaves still unanswered one crucial question: whether the preeminence of product innovations vis-a-vis process innovations rises along the growth path. The next proposition addresses this question. In particular, Proposition 5 shows that product innovations become increasingly prevalent along the growth, when considering a long enough time horizon.

Proposition 5 *Consider an economy that satisfies the condition $\sigma \geq \widetilde{\sigma}$, and which will therefore be able to sustain positive growth in the long run.*

Let $\mathcal{H}_T$=$\{1, 2, ..., T\}$ denote the set of periods starting in $t = 1$ and ending in $t = T$, and suppose that T is a large number.

Let $\alpha_T \in (0, 1)$ denote the fraction of periods in $\mathcal{H}_T$ that feature an equilibrium with process innovation effort.

Let the time horizon of $\mathcal{H}_T$ be extended by $\Delta \geq 1$ additional periods, and denote the extended set of periods by $\mathcal{H}_{T+\Delta}$=$\{1, 2, ..., T + \Delta\}$. Then, the fraction of periods in $\mathcal{H}_{T+\Delta}$ that feature process innovation effort, $\alpha_{T+\Delta} \in (0, 1)$, will be smaller than in $\mathcal{H}_T$ for Δ large enough. In other words, there exists $\widetilde{\Delta} \geq 1$, such that $\alpha_{T+\Delta} < \alpha_T$ for any $\Delta \geq \widetilde{\Delta}$.

Proposition 5 states that equilibria exhibiting process innovation effort tend to occur less and less often along the growth path. This in turn means that periods where the equilibrium features product innovation effort will tend to be increasingly observed along the growth path. This last result then complements the dynamics described in Proposition 4 regarding the importance and interplay between process and product innovations. It shows that, while both types of innovations fulfil crucial roles in the model to sustain long-run growth, product innovations tend to become increasingly more prominent than process innovations as economies grow richer.

6 Concluding Remarks

We presented a model where the combined impact of process and product innovations steer the economy along a growth path featuring both quantity and quality expansion. At early stages of development, when willingness to pay for quality upgrading is low, growth must be driven by the cost-cutting effect of process innovations. However, an economy cannot rely exclusively on process innovations in order to achieve long-lasting growth, as their benefits tend to decrease as physical production keeps expanding, pushing individuals towards a state of relative satiation. Sustained growth necessitates thus that the economy becomes also able to generate product innovations as it moves along the development path, so as to overturn the tendency towards satiation. In addition, quality-upgrading innovations boost the incentives to keep expanding physical production. Therefore, while process innovations are necessary to turn product innovations sufficiently profitable, product innovations are able to regenerate profit prospects from further process innovations. This implicit feedback loop may keep growth alive in the long run.

Our model has restricted the consumption space to a very specific case: one single final good available in different quality versions, which are all perfect substitutes among each other. One important type of innovation effort that our model has then ruled out is that one that leads to a *horizontal* expansion in the set of final goods, as in Judd (1985), Romer (1990), and Grossman and Helpman (1991c, Ch.3), Young (1993). In principle, these types of innovations may also be able to keep growth alive in the long run. In particular, as profit prospects from cost-cutting innovations dwindle owing to decreasing marginal utility in a *given* good category, individuals may at some point find it worthwhile to introduce a completely *new* good category. This new final good would offer initially large profit prospects from process innovations, which would tend to diminish with subsequent rounds of it. We see this mechanism leading to a horizontal expansion of the set of consumption goods as complementary to the interplay between quantity and quality expansion studied by our model. Certainly, a model in which growth features a simultaneous expansion in quantity, quality and variety of consumption, with positive feedbacks between all three dimensions, could yield a more encompassing description of growth in mature economies, and we see this as an appealing avenue of future research.

One other important limitation of the model is that firms and innovators are assumed to be alive for one period of time only. This simplification has two important implications. First, it cancels the effect that intertemporal externalities resulting from the interaction between process and product innovation could have on the decision to innovate in a given moment in time. Second, it removes any strategic consideration regarding creative destruction. Allowing for the positive effect of intertemporal externalities across types of innovation should help making the no-growth trap less likely to arise. On the other hand, the possibility of creative destruction may have the opposite effect. Incorporating these two dynamic considerations into the model could thus lead to smoother results in terms of growth dynamics and regime switches, possibly more aligned with those observed empirically.

Finally, our model has remained within a closed-economy framework. As a consequence, a crucial question it cannot address is whether international trade may somehow help spur growth in poorer economies by inducing quality-upgrading innovations there in order to cater to consumers located in richer economies.[20] On the other hand, even within a context with open economies, the presence of trade costs, combined with non-homothetic preferences, may as well lead to a *home-market effect* similar to that one featured in Fajgelbaum, Grossman and Helpman (2011) in a static framework, which could in turn push for dynamics in the opposite direction. In particular, a home-market effect stemming from nonhomothetic demand schedules may lead to dynamics where process innovation effort tends to gradually move to lower-income economies, while richer economies tend to increasingly specialize in generating product innovations. Studying the conditions under which either of these two conflicting forces seems to be the one that prevails would require a fully-fledged endogenous growth model with open economies, which is a task we leave open for future research.

[20] The closed-economy framework is also a limitation when considering cases of economies whose growth strategy has strongly relied on international trade, as the case of China in the past decade.

Appendix A: Proofs

Proof of Lemma 1. A household in t will optimally set $\varepsilon_t^* = 1$ if and only if the utility obtained from consuming $1+\sigma A_{t-1}+\sigma$ units of $q=1$ is strictly greater than the utility derived from consuming $1+\sigma A_{t-1}$ units of it plus the utility of leisure, η. Using (8), together with the utility function (5) when $Q_t = \{1\}$, condition (9) obtains. ■

Proof of Proposition 1. Consider some generic period $t = \tau - 1 \geq 1$, such that $A_{\tau-2} \leq \overline{A}(\sigma)$. Notice then that the equilibrium in $t = \tau - 1$ must feature $\varepsilon_{\tau-1}^* = 1$. Furthermore, $\varepsilon_t^* = 1$ must have also held true for all $t < \tau - 1$. Based on this, consider then a continuous sequence of equilibria with positive process innovation before τ, (1) implies that $A_{\tau-1} = \tau - 1$. Therefore, if $\tau - 1 \leq \overline{A}(\sigma)$, where $\overline{A}(\sigma)$ is given by (10), it follows from Lemma 1 that the equilibrium in period τ will feature $\varepsilon_\tau^* = 1$. On the other hand, if $\tau - 1 > \overline{A}(\sigma)$, Lemma 1 implies that the equilibrium in period τ will feature $\varepsilon_\tau^* = 0$. ■

Proof of Lemma 2. We carry out the proof in three separate steps.

Step 1) $P_t > (1+\rho)\, Y_t^{\rho/(1+\rho)}$ cannot hold in equilibrium.

Using (5) we can observe that the utility obtained by the household alive in t if they chose to consume the version of the good with quality $q = 1+\rho$ would be given by:

$$U_t(q = 1+\rho) = \ln \left[(1+\rho)\, \frac{Y_t}{P_t} \right]^{1+\rho}. \tag{20}$$

Instead, if they chose to consume the baseline quality version, they would obtain:

$$U_t(q = 1) = \ln (Y_t)\,. \tag{21}$$

Comparing (20) and (21), we can observe that $P_t > (1+\rho)\, Y_t^{\rho/(1+\rho)}$ implies $U_t(q = 1+\rho) < U_t(q = 1)$, and therefore the household would not consume the higher-quality version.

Step 2) $P_t < (1+\rho)\, Y_t^{\rho/(1+\rho)}$ cannot hold in equilibrium.

Suppose in equilibrium the firm that hires R_t requests the researcher to exert product innovation effort, and that it sets $P_t = \widetilde{P}_t < (1+\rho)\, Y_t^{\rho/(1+\rho)}$. In equilibrium, firms must necessarily satisfy the zero-profit condition. Hence, it must be the case that R_t is paid a remuneration equal to

$$\widetilde{\pi}_t = \widetilde{P}_t \frac{1+\sigma A_{t-1}}{1+\rho} - (1+\sigma A_{t-1})\,.$$

Suppose now that some other active firm belonging to $\mathcal{F}_t$ decided to offer R_t a remuneration $\widehat{\pi}_t$, where

$$\widehat{\pi}_t \equiv (\widetilde{P}_t + \widehat{\varepsilon}) \frac{1+\sigma A_{t-1}}{1+\rho} - (1+\sigma A_{t-1})\,, \quad \text{and} \quad \widehat{\varepsilon} > 0.$$

This firm would then attract R_t. Furthermore, this firm could charge a price $P_t' \equiv \widetilde{P}_t + \varepsilon' < (1+\rho)\, Y_t^{\rho/(1+\rho)}$, where $\varepsilon' > \widehat{\varepsilon} > 0$, for the higher-quality version of the final good, obtaining thus a positive profit. As a consequence, a situation where a firm charges a price $P_t < (1+\rho)\, Y_t^{\rho/(1+\rho)}$ for $q = 1+\rho$ while it also satisfies the zero profit condition cannot arise in equilibrium.

Step 3) Using again (20) and (21), we can first observe that when $P_t = (1+\rho)\, Y_t^{\rho/(1+\rho)}$ the household alive in t is indifferent between the baseline quality and the higher-quality version. Moreover, when (11) holds, there exist no profitable deviation to any firm $f_{k,t} \in \mathcal{F}_t$. In particular, in order to outcompete a firm whose strategy is characterized by (12) and (11), another firm should either offer the higher-quality version for a lower price or, alternatively, offer R_t a higher remuneration while keeping $P_t = (1+\rho)\, Y_t^{\rho/(1+\rho)}$ (since $P_t > (1+\rho)\, Y_t^{\rho/(1+\rho)}$ cannot hold in equilibrium, as shown by *Step 1*). Both deviations lead, however, to a loss. ■

Proof of Lemma 3. Using (11), we obtain that when R_t is hired to exert product innovation effort:

$$Y_t = P_t \frac{1+\sigma A_{t-1}}{1+\rho}. \tag{22}$$

Replacing (22) into (12) yields $P_t = (1+\rho) \left[P_t \left(1+\sigma A_{t-1}\right)(1+\rho)^{-1}\right]^{\frac{\rho}{1+\rho}}$, from where we may solve for P_t and obtain:

$$P_t = (1+\rho)\left(1+\sigma A_{t-1}\right)^{\rho}. \tag{23}$$

Lastly, plugging (23) into (22) yields (13). ■

Proof of Lemma 4. Part *i*). First of all, notice that using (11) and (13), it follows that when all R_t exerts product innovation effort, the level of utility achieved by the household alive in t will be given by

$$\begin{aligned} U_t\left(\varepsilon_{t,q} = 1 \middle|\, \omega_t = 1+\sigma A_{t-1}, r_t = \left(1+\sigma A_{t-1}\right)^{1+\rho} - \omega_t, P_t\right) \\ = \ln\left[(1+\rho)\, \frac{\left(1+\sigma A_{t-1}\right)^{1+\rho}}{P_t}\right]^{1+\rho}, \end{aligned} \tag{24}$$

Plugging (12), together with $Y_t = \left(1+\sigma A_{t-1}\right)^{1+\rho}$, into (24) yields:

$$U_t(\varepsilon_{t,q} = 1) = (1+\rho)\ln\left(1+\sigma A_{t-1}\right). \tag{25}$$

Suppose now this household would deviate from $\varepsilon_{t,q} = 1$, within that context, to $\varepsilon_t = 0$. In this case, $Y_t(\varepsilon_t = 0) = \omega_t = 1+\sigma A_{t-1}$. Recall that the price (12) leaves indifferent a household with $Y_t = \left(1+\sigma A_{t-1}\right)^{1+\rho}$ between the two versions of the consumption good. Since, $1+\sigma A_{t-1} < \left(1+\sigma A_{t-1}\right)^{1+\rho}$, it must then be the case that, if setting $\varepsilon_t = 0$, the household alive at time t will then strictly prefer consuming the baseline quality rather than (the more expensive) higher-quality version, $q = 1 + \rho$. Moreover, when $\omega_t = 1 + \sigma A_{t-1}$, there will always be some firm belonging to $\mathcal{F}_t$ willing to offer the baseline quality version, as it would break even by doing so. Hence, when the household alive in t sets $\varepsilon_t = 0$, within the described context, it will achieve as utility:

$$U_t\left(\varepsilon_t = 0 \middle| \omega_t = 1 + \sigma A_{t-1}, r_t = (1 + \sigma A_{t-1})^{1+\rho} - \omega_t, P_t = (1+\rho)(1 + \sigma A_{t-1})^{\rho}\right) \qquad (26)$$
$$= \ln(1 + \sigma A_{t-1}) + \eta$$

Finally, comparing (25) and (26), condition (14) ensures that $U_t(\varepsilon_{t,q} = 1| \cdot) > U_t(\varepsilon_t = 0| \cdot)$, completing the proof.

Part *ii*) Notice first that a firm belonging to $\mathcal{F}_t$ must offer at least a remuneration equal to $r_t = (1 + \sigma A_{t-1})^{1+\rho} - \omega_t$, with $\omega_t = 1 + \sigma A_{t-1}$, if it wishes to hire R_t and request R&D effort directed to process innovation from him. Hence, the surplus a firm belonging to $\mathcal{F}_t$ would obtain from requesting process innovation effort from R_t would be equal to $\sigma - r_t$, which using $r_t = (1 + \sigma A_{t-1})^{1+\rho} - \omega_t$ and $\omega_t = 1 + \sigma A_{t-1}$, yields $\sigma - r_t = (1 + \sigma A_{t-1} + \sigma) - (1 + \sigma A_{t-1})^{1+\rho}$. Given that (15) implies $(1 + \sigma A_{t-1} + \sigma) - (1 + \sigma A_{t-1})^{1+\rho} < 0$, no firm active in t would, in this context, prefer to deviate to requesting process innovation effort from R_t. ■

Proof of Proposition 2. Notice first that in an equilibrium with $\varepsilon^*_{t,q} = 1$, the remuneration received by R_t will be equal to $(1 + \sigma A_{t-1})^{1+\rho} - (1 + \sigma A_{t-1})$. Notice also that, when $q_{t-1} = 1$, the surplus generated by a process innovation is equal to σ. As a result, if $\sigma > (1 + \sigma A_{t-1})^{1+\rho} - (1 + \sigma A_{t-1})$, an equilibrium with $\varepsilon^*_{t,q} = 1$ cannot arise. On the other hand, if $\sigma < (1 + \sigma A_{t-1})^{1+\rho} - (1 + \sigma A_{t-1})$, the non-deviation condition (15) will be verified in an equilibrium with $\varepsilon^*_{t,q} = 1$. Furthermore, $\sigma < (1 + \sigma A_{t-1})^{1+\rho} - (1 + \sigma A_{t-1})$ implies that an equilibrium with $\varepsilon^*_{t,p} = 1$ cannot arise either, as the deviation to $\varepsilon^*_{t,q} = 1$ becomes profitable to firms. Secondly, notice that using (12) and (13), from (5) it follows that the utility of the household alive in t when $\varepsilon^*_{t,q} = 1$ will be given by $U_t(\varepsilon^*_{t,q} = 1) = (1 + \rho)\ln(1 + \sigma A_{t-1}) - \eta$, while the utility it would obtain if it deviated to $\varepsilon_t = 0$ would be $U_t(\varepsilon_t = 0) = \ln(1 + \sigma A_{t-1})$. Comparing these two expressions, we can observe that (14) implies that such a deviation would not be profitable, while if it that condition failed to hold the household would rather set $\varepsilon_t = 0$. Finally, from a situation in which $\varepsilon_t = 0$, a firm could offer a remuneration to R_t high enough to induce the household to sell their R&D time endowment to the firm, which would in turn use it for product innovation effort so long as (14) holds true. Hence, there cannot exist an equilibrium with $\varepsilon^*_t = 0$ when (14) holds true. ■

Proof of Lemma 5. Equalizing (10) and (16), yields to $\overline{A}(\sigma) = \underline{A}_1(\sigma)$ if and only if $\sigma = \widehat{\sigma} \equiv e^{\eta/\rho}(e^{\eta} - 1)$. Next, plugging this value into either (10) or (16) yields:

$$\overline{A}(\widehat{\sigma}) = \underline{A}_1(\widehat{\sigma}) = \frac{e^{\eta/\rho} - 1}{e^{\eta/\rho}(e^{\eta} - 1)}. \tag{27}$$

Finally, plugging $\sigma = \widehat{\sigma}$ and $\underline{A}_2 = \left(e^{\eta/\rho} - 1\right)/[e^{\eta/\rho}(e^{\eta} - 1)]$ into the LHS of (17), one can verify that equation (17) holds true for those values of σ and $\underline{A}_2$. As a result, it follows that $\overline{A}(\widehat{\sigma}) = \underline{A}_1(\widehat{\sigma}) = \underline{A}_2(\widehat{\sigma}) = \underline{A}(\widehat{\sigma})$, where $\widehat{\sigma} = e^{\eta/\rho}(e^{\eta} - 1)$. In addition, given that $\overline{A}'(\sigma) > 0$ and $\underline{A}'(\sigma) < 0$, it also follows that $\overline{A}(\sigma) \gtreqless \underline{A}(\sigma)$ whenever $\sigma \gtreqless \widehat{\sigma}$. ■

Proof of Proposition 3. Notice first that, whenever $\sigma < \widehat{\sigma}$, using (16) and (17) we have that $\underline{A}(\sigma) = \underline{A}_1$. Hence, using (10) and (16), we can observe that whenever $\sigma < \widehat{\sigma}$, we have that $\underline{A}(\sigma) - \overline{A}(\sigma) = (e^{\eta/\rho}/\sigma) - (e^{\eta} - 1)^{-1}$. Since $\bar{t} \equiv \text{integer}\left\{1 + \overline{A}(\sigma)\right\} \leq 1 + \overline{A}(\sigma)$, then the economy will not be able to switch to an equilibrium with product innovation when $(e^{\eta/\rho}/\sigma) - (e^{\eta} - 1)^{-1} \geq 1$, from where it follows that a sufficient condition for this switch not to take place is that $\sigma \leq e^{\eta/\rho}(1 - 1)e^{-\eta}$. Since $\sigma \geq \widehat{\sigma}$ is a sufficient condition for $A_{\bar{t}} > \underline{A}(\sigma)$, a regime switch would take place no later than in $\bar{t} + 1$ when $\sigma \geq \widehat{\sigma}$ holds true. As a consequence, by continuity, there must exist some $\widetilde{\sigma}$, such that $e^{\eta/\rho}(1 - 1)e^{-\eta} < \widetilde{\sigma} < \widehat{\sigma}$, below which the economy will never reach a situation where $A_{t-1} \geq \underline{A}(\sigma)$, and the regime-switch will never take place along the growth path. By using the same reasoning, it follows as well that when σ lies above $\widetilde{\sigma}$, we have that $\bar{t} \equiv \text{integer}\left\{1 + \overline{A}(\sigma)\right\} > \underline{A}(\sigma)$, and the economy will thus be able to switch at some point to an equilibrium with product innovation effort in a period no later than $t = \bar{t} + 1$, given by $\underline{t} + 1$, where $\underline{t} \equiv \text{integer}\left\{1 + \underline{A}(\sigma)\right\}$. ■

Proof of Lemma 6. First of all, notice that when $\sigma \geq \widetilde{\sigma}$, Proposition 3 implies that, in equilibrium, the economy will experience process innovation effort during $t \leq \underline{t}$, and product innovation effort at $t = \underline{t} + 1$. The rest of the proof will show that for any $t' > \underline{t} + 1$ the action $\varepsilon_{t',q} = 1$ will dominate $\varepsilon_{t'} = 0$.

Step 1. Generalization of Lemma 2 for any $q \in Q$: The expected return of the product innovation blueprint designed by a generic R_t alive in t will be:

$$\pi_t^q = P_t(q_t)\frac{1 + \sigma A_t}{q_t} - \omega_t, \tag{28}$$

where $P_t(q_t)$ is the price of the (newly designed) quality version $q_t \in Q$. To compute the equilibrium value of $P_t(q_t)$, notice that, based on (5), this will follow from the condition

$$\ln\left(q_t \frac{Y_t}{P_t(q_t)}\right)^{q_t} \geq \ln(Y_t), \tag{29}$$

from where we can obtain

$$P_t(q_t) = q_t \, Y_t^{(q_t-1)/q_t} \tag{30}$$

when (29) holds with equality. Lastly, the fact that all firms active in t inherit a technology that allows producing $(1+\sigma A_{t-1})/q_{t-1}$ units of the quality version $q_{t-1} \in Q$, where $q_{t-1} = q_t - \rho$, with one unit of labor in turn implies that:

$$\omega_t = P_t(q_t)\frac{1+\sigma A_{t-1}}{q_{t-1}} = Y_t^{(q_{t-1}-1)/q_{t-1}} \left(1+\sigma A_{t-1}\right). \tag{31}$$

Step 2. Generalization of Lemma 3 for any $q \in Q$: Consider an equilibrium in a generic time t where $\varepsilon^*_{t,q} = 1$. Then using (28), (30) and (31), we obtain:

$$Y_t(\varepsilon_{t,q} = 1) = (1+\sigma A_{t-1})^{q_t}. \tag{32}$$

Step 3. Consider now some moment in time $t' > \underline{t}+1$. Due to Proposition 3, when $\sigma \geq \widetilde{\sigma}$, we must have that $A_{t'-1} \geq \underline{t} \equiv \text{integer}\{\underline{A}(\sigma)+1\}$ and $q_{t'-1} \geq 1+\rho$. Suppose also that $\varepsilon_{t,q} = 1$ in each period $t \geq \underline{t}+1$. Then, using (32):

$$Y_{t'}(\varepsilon_{t',q} = 1) = (1+\sigma \underline{t})^{q_{t'}}. \tag{33}$$

Using next the utility function (5), together with (33) and (30), we may obtain:

$$U_{t'}(\varepsilon_{t',q} = 1) = q_{t'} \ln(1+\sigma\underline{t}), \tag{34}$$

which denotes the level of utility achieved by a generic household alive in $t = t'$ when $\varepsilon_{t,q} = 1$ for all $t \geq \underline{t}+1$. On the other hand, if in such same circumstance $R_{t'}$ deviates to $\varepsilon_{t'} = 0$, the household alive in t' will achieve a utility level given by:

$$U_{t'}(\varepsilon_{t'} = 0) = (q_{t'} - \rho)\ln(1+\sigma\underline{t}) + \eta. \tag{35}$$

Comparing (34) and (35), we obtain:

$$U_{t'}(\varepsilon_{t',q} = 1) > U_{t'}(\varepsilon_{t'} = 0) \iff \rho \ln(1+\sigma\underline{t}) > \eta. \tag{36}$$

But now notice that $\rho \ln(1+\sigma\underline{t}) > \eta$ actually holds true for any economy that verifies $\sigma > \widetilde{\sigma}$, completing the proof. ■

Proof of Proposition 4.

Part 1. The fact that there is an initial growth phase, between $t = 1$ and $t = \underline{t} \geq 1$ follows from Proposition 3.

Part 2. The fact that there is a second growth phase starting in $t = \underline{t} + 1$ that is driven by product innovation effort also follows from Proposition 3. Next, to prove that this growth phase lasts for a finite number of periods, we proceed by contradiction. Suppose that the only type of innovation effort undertaken during $t > \underline{t} + 1$ is in product innovation. In this case, we would have $A_{t-1} = \underline{t}$ for all $t > \underline{t} + 1$. In addition, the highest quality version available in periods $t > \underline{t} + 1$ would be $q_t = 1 + (t - \underline{t})\rho$. As a consequence, when $\varepsilon_{t,q} = 1$ for all $t > \underline{t} + 1$, the expected return generated by a product innovation blueprint (net of wages, ω_t) will be

$$\pi_t^q(\widetilde{\varepsilon}_{q,t>\underline{t}+1}) = (1 + \sigma\underline{t})^{1+(t-\underline{t})\rho} - \omega_t, \tag{37}$$

where we use $\widetilde{\varepsilon}_{q,t>\underline{t}+1}$ to denote the *hypothetical* path in which $\varepsilon_{t,q} = 1$ for all $t > \underline{t} + 1$. For this to be an equilibrium, $\pi_t^q(\widetilde{\varepsilon}_{q,t>\underline{t}+1})$ should be larger than the expected return obtained when shifting R_t alive in a generic $t > \underline{t} + 1$ to setting $\varepsilon_{t,p} = 1$, which would yield:

$$\pi_t^p(\varepsilon_{t,p} = 1) = Y_t^{(t-1-\underline{t})\rho/[1+(t-1-\underline{t})\rho]} (1 + \sigma\underline{t} + \sigma) - \omega_t. \tag{38}$$

Comparing (37) and (38), we can observe that $\pi_t^q(\widetilde{\varepsilon}_{q,t>\underline{t}+1}) > \pi_t^p(\varepsilon_{i,t,p} = 1)$ requires $(1 + \sigma\underline{t})^{1+(t-\underline{t})\rho} \geq (1 + \sigma\underline{t} + \sigma)^{1+[(t-1)-\underline{t}]\rho}$, which will fail to hold true when t becomes sufficiently large (i.e., when t departs sufficiently from $\underline{t}$).

Part 3. Lastly, to prove that after $t = \widehat{t} \geq \underline{t} + 1$ the economy will be able to sustain positive growth forever by alternating finite spells where the equilibrium features process innovation effort with finite spells where the equilibrium features product innovation effort, we proceed by again contradiction, while bearing in mind the result in Lemma 6. Consider first the case of a hypothetical economy that for all periods $t \geq t'$ features process innovation effort in equilibrium, where we let $t' > \widehat{t}$. In that case, we will have $A_t = t - (q_{t'-1} - 1)/\rho$ and $q_t = q_{t'-1}$, for all $t \geq t'$. The expected returns of a process innovation blueprint in given period $t > t'$ will be

$$\pi_t^p(\widetilde{\varepsilon}_{p,t\geq t'}) = \{1 + \sigma[t - (q_{t'-1} - 1)/\rho]\}^{q_{t'-1}} - \omega_t, \tag{39}$$

where $\widetilde{\varepsilon}_{p,t\geq t'}$ denotes the *hypothetical* growth path in which $\varepsilon_{t,p} = 1$ for all $t \geq t'$. For this to be an equilibrium, $\pi_t^p(\widetilde{\varepsilon}_{p,t>\underline{t}+1})$ should be larger than the expected return obtained when shifting researcher R_t alive in a generic $t \geq t'$ to setting $\varepsilon_{t,q} = 1$, which would yield

$$\pi_t^q(\varepsilon_{t,q} = 1) = Y_t^{(q_{t'-1}+\rho-1)/(q_{t'-1}+\rho)} \{1 + \sigma[t - 1 - (q_{t'-1} - 1)/\rho]\} - \omega_t. \tag{40}$$

Comparing (39) and (40), it follows that $\pi_t^p(\widetilde{\varepsilon}_{p,t\geq t'}) \geq \pi_t^q(\varepsilon_{t,q}=1)$ requires

$$\{1+\sigma\left[t-(q_{t'-1}-1)/\rho\right]\}^{q_{t'-1}} > \{1+\sigma\left[t-1-(q_{t'-1}-1)/\rho\right]\}^{q_{t'-1}+\rho}$$

which will fail to hold when t becomes sufficiently large. As a consequence, the economy cannot possibly sustain an equilibrium growth path with only process innovation effort over an infinitely long sequence of consecutive periods.

Consider next the case of a hypothetical economy that for all periods $t \geq t'$ features product innovation effort in equilibrium, where again we let $t' > \widehat{t}$. In this case, $A_{t-1} = (t'-1) - (q_{t'-1}-1)/\rho$ and $q_t = q_{t'-1} + (t-t'+1)\rho$, for all $t \geq t'$. These results in turn imply that in a generic period $t \geq t'$ we will have:

$$\pi_t^q(\widetilde{\varepsilon}_{q,t\geq t'}) = \{1+\sigma\left[(t'-1)-(q_{t'-1}-1)/\rho\right]\}^{q_{t'-1}+(t-t'+1)\rho} - \omega_t, \tag{41}$$

where we use $\widetilde{\varepsilon}_{q,t\geq t'}$ to denote the *hypothetical* path in which $\varepsilon_{t,q}=1$ for all $t \geq t'$. For this to be an equilibrium, $\pi_t^q(\widetilde{\varepsilon}_{q,t>\underline{t}+1})$ should be larger than the expected return obtained when shifting researcher R_t alive in a generic $t \geq t'$ to setting $\varepsilon_{t,p}=1$, which would yield:

$$\pi_t^p(\varepsilon_{t,p}=1) = Y_t^{[q_{t'-1}+(t-t')\rho-1]/[q_{t'-1}+(t-t')\rho]}\{1+\sigma\left[t'-(q_{t'-1}-1)/\rho\right]\} - \omega_t. \tag{42}$$

Comparing (41) and (42), it follows that $\pi_t^q(\widetilde{\varepsilon}_{q,t\geq t'}) > \pi_t^p(\varepsilon_{i,t,p}=1)$ requires

$$\{1+\sigma\left[(t'-1)-(q_{t'-1}-1)/\rho\right]\}^{q_{t'-1}+(t-t'+1)\rho} \geq \{1+\sigma\left[t'-(q_{t'-1}-1)/\rho\right]\}^{q_{t'-1}+(t-t')\rho}$$

which will fail to hold when t becomes sufficiently large. As a result, the economy cannot possibly sustain an equilibrium growth path with only product innovation effort over an infinitely long sequence of consecutive periods.

The previous two contradictions imply thus that there cannot exist an equilibrium growth path featuring either infinitely long spells of process innovation effort or infinitely long spells of product innovation effort. Lemma 6 stipulates that an economy satisfying $\sigma \geq \widetilde{\sigma}$ will always able to sustain an equilibrium with some type of innovation effort. Hence, it must be the case that the growth path will necessarily exhibit finite spells with process innovation effort in equilibrium, alternating with finite spells with product innovation effort in equilibrium. ■

Proof of Proposition 5. When T is a large number and the fraction of periods in $\mathcal{H}_T$ featuring an equilibrium with $\varepsilon_{t,p}^*=1$ is given by α_T, we have that:

$$q_T = 1+(1-\alpha_T)\,T\rho \tag{43}$$

$$A_T = \alpha_T\,T. \tag{44}$$

Using (43) and (44), we can observe that the equilibrium in $T+1$ will feature $\varepsilon^*_{T+1,p} = 1$ if $(1+\alpha_T T\sigma)^{1+(1-\alpha_T)T\rho+\rho} < (1+\alpha_T T\sigma+\sigma)^{1+(1-\alpha_T)T\rho}$ holds true, while the economy will have an equilibrium with $\varepsilon^*_{T+1,q} = 1$ when $(1+\alpha_T T\sigma)^{1+(1-\alpha_T)T\rho+\rho} > (1+\alpha_T T\sigma+\sigma)^{1+(1-\alpha_T)T\rho}$. The previous two inequalities can be combined, to obtain

$$\begin{aligned} \varepsilon^*_{T+1,q} &= 1 \quad \text{if} \quad \Gamma(\alpha_T, T) > \Upsilon(\alpha_T, T), \\ \varepsilon^*_{T+1,p} &= 1 \quad \text{if} \quad \Gamma(\alpha_T, T) < \Upsilon(\alpha_T, T). \end{aligned} \tag{45}$$

where

$$\Gamma(\alpha_T, T) \equiv 1 + \frac{\rho}{1+(1-\alpha_T)\,T\rho} \quad \text{and} \quad \Upsilon(\alpha_T, T) \equiv \frac{\ln(1+\alpha_T T\sigma+\sigma)}{\ln(1+\alpha_T T\sigma)}. \tag{46}$$

Notice, first, from (46) that $\partial\Gamma(\cdot)/\partial\alpha_T > 0, \partial\Upsilon(\cdot)/\partial\alpha_T < 0$, $\partial\Gamma(\cdot)/\partial T < 0$, and $\partial\Upsilon(\cdot)/\partial T < 0$. It can also be shown (see proof in Appendix B) that, there exists a finite value $\widehat{T}(\alpha_T)$, such that $\Gamma(\alpha_T, \widehat{T}(\alpha_T)) = \Upsilon(\alpha_T, \widehat{T}(\alpha_T))$, and $\Gamma(\alpha_T, T) < \Upsilon(\alpha_T, T)$ for $T < \widehat{T}$ while $\Gamma(\alpha_T, T) > \Upsilon(\alpha_T, T)$ for $T > \widehat{T}$.

Suppose now that after extending the horizon from T to $T+\Delta$, the fraction of periods in which the equilibrium exhibits $\varepsilon^*_{t,p} = 1$ remains equal to α_T. Then, there will be some $\widetilde{\Delta}$ such that for all $\Delta > \widetilde{\Delta}$, we will have that $\Gamma(\alpha_T, T+\Delta) > \Upsilon(\alpha_T, T+\Delta)$. As a consequence, the fraction of periods in which $\varepsilon^*_{t,p} = 1$ cannot remain indefinitely equal to α_T. Furthermore, since $\partial\Gamma(\cdot)/\partial\alpha_T > 0$ and $\partial\Upsilon(\cdot)/\partial\alpha_T < 0$, it must then be the case that $\alpha_{T+\Delta} < \alpha_T$ for $\Delta > \widetilde{\Delta}$, where $\widetilde{\Delta}$ is a finite positive value. ■

Appendix B: Additional Proofs

Proof that the threshold $\underline{A}_2$ is a decreasing function of σ.

Note that an alternate way to implicitly define $\underline{A}_2$ is by applying logs on (17). This leads to:

$$\underbrace{(1+\rho)\ln(1+\sigma\underline{A}_2)-\ln(1+\sigma\underline{A}_2+\sigma)}_{\Psi(\rho,\sigma,\underline{A}_2)}=0. \tag{47}$$

From (47) we can observe several important properties of the function $\Psi(\rho,\sigma,\underline{A}_2)$: *i)* $\Psi'_{\underline{A}_2}(\cdot)>0$, *ii)* $\Psi'_{\rho}(\cdot)>0$, *iii)* $\Psi(\rho,\sigma,\underline{A}_2=0)=-\ln(1+\sigma)<0$, *iv)* $\lim_{\underline{A}_2\to\infty}\Psi(\rho=0,\sigma,\underline{A}_2)=0$. A first result to notice is that since $\Psi(\rho=0,\sigma,\underline{A}_2=\infty)=0$ for any $\sigma>0$, and we have that $\Psi'_{\rho}(\cdot)>0$, it must then be that for any $\rho>0$ there is a unique, finite and strictly positive value of $\underline{A}_2$ such that it satisfies $\Psi(\rho,\sigma,\underline{A}_2)=0$. Furthermore, combining this with $\Psi'_{\underline{A}_2}(\cdot)>0$, it follows that $\partial\underline{A}_2/\partial\rho<0$. Using now the full expression $\Psi'_{\rho}(\cdot)=\ln(1+\sigma\underline{A}_2)$, we can also see that $\partial\Psi'_{\rho}(\cdot)/\partial\sigma>0$. Since $\Psi(\rho=0,\sigma,\underline{A}_2=\infty)=0$ for any $\sigma>0$, and $\Psi'_{\rho}(\cdot)>0$, it must then be the case that, considering two generic $\underline{\sigma}<\overline{\sigma}$ and letting $\Psi(\rho,\underline{\sigma},\underline{A}_2(\underline{\sigma}))=0$ and $\Psi(\rho,\overline{\sigma},\underline{A}_2(\overline{\sigma}))=0$, we must have $\underline{A}_2(\overline{\sigma})<\underline{A}_2(\underline{\sigma})$. ■

Proof of Existence of $\widehat{T}(\boldsymbol{\alpha}_T)$. Let first define $\Lambda(\cdot)\equiv\Gamma(\cdot)-\Upsilon(\cdot)$. Thus,

$$\Lambda(T)\equiv\frac{1+(1-\alpha)\rho T+\rho}{1+(1-\alpha)\rho T}-\frac{\ln(1+\alpha\sigma T+\sigma)}{\ln(1+\alpha\sigma T)}, \tag{48}$$

and notice that $\lim_{T\to0}\Lambda(T)=-\infty$ and $\lim_{T\to\infty}\Lambda(T)=0$. Differentiating (48) with respect to T yields:

$$\Lambda'(T)=-\frac{\rho^2(1-\alpha)}{[1+(1-\alpha)\rho T]^2}-\alpha\sigma\frac{\dfrac{\ln(1+\alpha\sigma T)}{1+\alpha\sigma T+\sigma}-\dfrac{\ln(1+\alpha\sigma T+\sigma)}{1+\alpha\sigma T}}{[\ln(1+\alpha\sigma T)]^2}. \tag{49}$$

Computing the limits of (49) as $T\to0$ and $T\to\infty$ yields, respectively: $\lim_{T\to0}\Lambda'(T)=+\infty$ and $\lim_{T\to\infty}\Lambda'(T)=0$. Hence, the function $\Lambda(\cdot)$ approaches asymptotically $\Lambda(T)=0$ as $T\to\infty$, while $\Lambda(\cdot)<0$ for sufficiently low levels of T. Next, we state and prove two auxiliary results to build the proof that $\Lambda(\cdot)$ will necessarily become positive for some finite level of $T=\widehat{T}>0$, and remain positive for $T>\widehat{T}$, approaching thus $\Lambda(T)=0$ as $T\to\infty$ from above.

Lemma A1 *There exists a positive and finite value $T=\widetilde{T}$ such that $\Lambda'(\widetilde{T})=0$.*

Proof. *Equalizing the expression in (49) to zero leads to*

$$\Psi(T)\equiv\frac{\dfrac{\ln(1+\alpha T\sigma+\sigma)}{1+\alpha T\sigma}-\dfrac{\ln(1+\alpha T\sigma)}{1+\alpha T\sigma+\sigma}}{[\ln(1+\alpha T\sigma)]^2}[1+(1-\alpha)T\rho]^2=\frac{\rho^2(1-\alpha)}{\alpha\sigma}. \tag{50}$$

From the LHS of (50) notice first that $\lim_{T\to 0}\Psi(T)=\infty$*. Moreover, computing* $\lim_{T\to\infty}\Psi(T)$*:*

$$\lim_{T\to\infty}\Psi(T)=\underbrace{\lim_{T\to\infty}\left[\frac{1}{\ln(1+\alpha T\sigma)}\right]}_{\text{converges to }0}\times\underbrace{\lim_{T\to\infty}\left[\frac{\ln(1+\alpha T\sigma+\sigma)}{\ln(1+\alpha T\sigma)}\frac{\left[1+(1-\alpha)\,T\rho\right]^2}{(1+\alpha T\sigma)}-\frac{\left[1+(1-\alpha)\,T\rho\right]^2}{(1+\alpha T\sigma+\sigma)}\right]}_{\text{converges to }\frac{(1-\alpha)^2\rho^2}{\alpha^2\sigma}}=0.$$

Therefore, since $\frac{\rho^2(1-\alpha)}{\alpha\sigma}>0$*, by continuity there must then exist some positive and finite value* $T=\widetilde{T}$ *such that* $\Lambda'(\widetilde{T})=0$*.* ♦

Lemma A2 For T sufficiently large $\Lambda(T)>0$. In other words, $\lim_{T\to\infty}\Lambda(T)=0^+$.

Proof. Note from (48) that $\Lambda(T)>0$ if and only if

$$\frac{(1+\alpha\sigma T)^{1+(1-\alpha)\rho T+\rho}}{(1+\alpha\sigma T+\sigma)^{1+(1-\alpha)\rho T}}>1. \tag{51}$$

Applying the limit as $T\to\infty$ on the RHS of (51):

$$\lim_{T\to\infty}\left[\frac{(1+\alpha\sigma T)^{1+(1-\alpha)\rho T+\rho}}{(1+\alpha\sigma T+\sigma)^{1+(1-\alpha)\rho T}}\right]=\lim_{T\to\infty}\left[\frac{(1+\alpha\sigma T)^{1+(1-\alpha)\rho T}}{(1+\alpha\sigma T+\sigma)^{1+(1-\alpha)\rho T}}\right]\times\lim_{T\to\infty}(1+\alpha\sigma T)^{\rho}.$$

Note first that $\lim_{T\to\infty}(1+\alpha\sigma T)^{\rho}=+\infty$. Secondly, let

$$\Xi\equiv\frac{(1+\alpha\sigma T)^{1+(1-\alpha)\rho T}}{(1+\alpha\sigma T+\sigma)^{1+(1-\alpha)\rho T}}. \tag{52}$$

Applying logs on the RHS of (52), it follows that:

$$\lim_{T\to\infty}\ln(\Xi)=\lim_{T\to\infty}\frac{\ln\left(\dfrac{1+\alpha\sigma T}{1+\alpha\sigma T+\sigma}\right)}{\left[1+(1-\alpha)\,\rho T\right]^{-1}}. \tag{53}$$

Next, using L'Hopital's rule to compute the limit in the RHS of (53) leads to:

$$\lim_{T\to\infty}\ln(\Xi)=-\frac{1-\alpha}{\alpha}\rho. \tag{54}$$

Hence, using finally the fact that $\lim_{T\to\infty}\ln(\Xi)=\ln\left[\lim_{T\to\infty}(\Xi)\right]$, it follows that $\lim_{T\to\infty}(\Xi)=e^{-\frac{1-\alpha}{\alpha}\rho}>0$. Therefore, it must be the case that (51) holds true for a sufficiently large value of T and $\lim_{T\to\infty}\Lambda(T)=0^+$. ♦

Merging the results of the previous two lemmas, plus the fact that $\lim_{T\to 0}\Lambda(T)=-\infty$ and $\lim_{T\to\infty}\Lambda(T)=0$, by continuity it follows there must exist some finite positive value $\widehat{T}<\widetilde{T}$, such that: *i)* $\Lambda(T)<0$ for all $0<T<\widehat{T}$, *ii)* $\Lambda(\widehat{T})=0$, and *iii)* $\Lambda(T)<0$ for all $T>\widehat{T}$. ■

Appendix C: A Version of the Model with Simultaneous Process and Product Innovation Effort

The benchmark model has worked under the assumption that firms are not able to split the researcher's time between process innovation effort and product innovation effort. In this appendix, we relax this assumption, by letting the firm that hires R_t set any combinations of $\varepsilon_{t,p} \in [0,1]$ and $\varepsilon_{t,q} \in [0,1]$, subject to the condition that $\varepsilon_{t,q} + \varepsilon_{t,p} = 1$.

When firms can split R_t's R&D effort between both $\varepsilon_{t,p}$ and $\varepsilon_{t,q}$, in an equilibrium with positive R&D effort, the exact solutions of $\varepsilon^*_{t,p}$ and $\varepsilon^*_{t,q}$ will stem from the firms' optimization problem, together with the zero-profit condit on R&D effort surplus. Letting $\lambda_t \in [0,1]$ denote the share of R_t's R&D effort time directed to process innovation effort (i.e., letting $\varepsilon_{t,p} = \lambda_t$ and $\varepsilon_{t,q} = 1 - \lambda_t$), it follows that, in an equilibrium with positive R&D effort, the value of λ^*_t will stem from the following optimization problem:[21]

$$\max_{\lambda_t \in [0,1]} : \widetilde{\pi}^{\varepsilon}_t \equiv (1 + \sigma A_{t-1} + \lambda_t \sigma)^{q_{t-1} + \rho(1-\lambda_t)}, \tag{55}$$

where now:

$$A_{t-1} = \sum\nolimits_0^{t-1} \varepsilon_t \times \lambda_\tau \quad \text{and} \quad q_{t-1} = 1 + \rho \sum\nolimits_0^{t-1} \varepsilon_t \times (1 - \lambda_\tau). \tag{56}$$

Equilibrium in t=1

When computing the equilibrim in $t = 1$, notice first that $A_0 = 1$ and $q_0 = 1$. As a result, in an equilibrium with positive R&D effort in $t = 1$, the equilibrium level λ^*_1 would stem from:

$$\max_{\lambda_1 \in [0,1]} : \widetilde{\pi}^{\varepsilon}_t = (1 + \lambda_1 \sigma)^{1 + \rho(1-\lambda_1)}. \tag{57}$$

For future reference we will now define:

$$\widetilde{\rho}(\sigma) \equiv \frac{\sigma}{1+\sigma} \frac{1}{\ln(1+\sigma)}, \tag{58}$$

[21] To see this, notice that the firm that hires R_t will seek to maximize the surplus generated by his R&D effort, which bearing in mind that $\omega_t = 1 + \sigma A_{t-1}$, is given by:

$$\pi^{\varepsilon}_t = P(q_t) \frac{1 + \sigma A_{t-1} + \lambda \sigma}{q_t} - (1 + \sigma A_{t-1}), \quad \text{with } q_t = q_{t-1} + (1-\lambda)\rho.$$

Given that, in equilibrium, $P(q_t) = q_t Y_t^{(q_t - 1)/q_t}$, and given that $Y_t = \pi^{\varepsilon}_t + \omega_t$, those expressions imply that $\pi^{\varepsilon}_t = (1 + \sigma A_{t-1} + \lambda\sigma)^{q_{t-1} + (1-\lambda)\rho} - (1 + \sigma A_{t-1})$. As a result, when the firm seeks to maximize π^{ε}_t with respect to $\lambda \in [0,1]$, it needs to solve problem (55), where notice that $\widetilde{\pi}^{\varepsilon}_t = \pi^{\varepsilon}_t - (1 + \sigma A_{t-1})$.

where notice that $\widetilde{\rho}'(\sigma) < 0$.

We can now state the following result, which extends the result that there cannot exist an equilibrium with product innovation effort in $t = 1$.

Result 1 *Consider the case of an economy in $t = 1$, such that it verifies Assumption 3 (i.e., $\sigma \geq e^{\eta} - 1$). Then, the equilibrium in $t = 1$ entails positive R&D effort. Furthermore:*

i) If $\rho \leq \widetilde{\rho}(\sigma)$, all the R&D effort is directed to process innovation effort; that is, $\lambda_1^* = 1$.

ii) If $\rho > \widetilde{\rho}(\sigma)$, R&D effort is split between process innovation effort and product innovation effort; that is, $\lambda_1^* \in (0, 1)$, and λ_1^* stems from:

$$\frac{\sigma}{\rho} \frac{1 + \rho(1 - \lambda_1^*)}{(1 + \lambda_1^* \sigma)} = \ln(1 + \lambda_1^* \sigma). \tag{59}$$

Proof. *Notice that solving (57), entails that an optimum with $\lambda_1^* < 1$, must satisfy:*

$$\sigma \frac{1 + \rho(1 - \lambda_1^*)}{(1 + \lambda_1^* \sigma)} - \rho \ln(1 + \lambda_1^* \sigma) \leq 0 \quad \textit{and} \quad \left[\sigma \frac{1 + \rho(1 - \lambda_1^*)}{(1 + \lambda_1^* \sigma)} - \rho \ln(1 + \lambda_1^* \sigma)\right] \times \lambda_1^* = 0. \tag{60}$$

On the other hand, an optimum with $\lambda_1^* = 1$ must satisfy:

$$\frac{\sigma}{(1 + \sigma)} - \rho \ln(1 + \sigma) \geq 0. \tag{61}$$

Firstly, notice that (61) holds true if and only if $\rho \leq \widetilde{\rho}(\sigma)$, where $\widetilde{\rho}(\sigma)$ is given by (58). Secondly, from (60), it can be observed that $\lambda_1^* = 0$ cannot hold true, as it would require $\sigma(1 + \rho) \leq 0$. Lastly, when $\rho > \widetilde{\rho}(\sigma)$, it must thus be that (60) will hold true in an interior point $\lambda_1^* \in (0, 1)$, pinned down by (59). ■

Result 1 essentially states that there cannot exist in $t = 1$ an equilibrium where all R&D effort is directed to product innovation (that is, $\lambda_1^* = 0$ cannot hold true in the equilibrium in $t = 1$). Furthermore, Result 1 also shows that when ρ lies below $\widetilde{\rho}(\sigma)$, the equilibrium in $t = 1$ will have $\lambda_1^* = 1$; that is, all R&D effort is directed to process innovation.

Product Innovation Effort in a Generic Period *t*

We will now restrict the attention to the case of an economy that satisfies $\rho \leq \widetilde{\rho}(\sigma)$, and hence exhibits $\lambda_1^* = 1$ in the equilibrium in $t = 1$. In addition to that, in order to focus on the most interesting dynamics of the model, we will assume that η is small enough (relative to σ and ρ) so as to ensure that the switch to an equilibrium with positive product innovation effort can actually take place along the growth path.[22] The next result shows that there will exist some $t > 1$ with positive product innovation effort.

Result 2 *Consider the case of an economy that undergoes a sufficiently long number of periods where $\lambda_t^* = 1$. Then, it must necessarily be the case that $\lambda_{t+1}^* < 1$ for t long enough.*

Proof. *Notice that if $\lambda_t^* = 1$ holds for t periods, then $\lambda_{t+1}^* = 1$ will require that*

$$\frac{1}{1 + \sigma t + \sigma} > \rho \ln(1 + \sigma t + \sigma),$$

which will necessarily fail to hold to t large enough. ■

Result 2 above essentially shows that, along a growth path of an the economy that starts off with $\lambda_1^* = 1$, this economy will eventually reach a point in time when it will also start using part of R_t's time endowment for product innovation effort. The intuition for this result is analogous to that one in the main model with the assumption of complete specialization of innovation effort by type. The main difference when R&D effort time can be split between $\varepsilon_{t,p}$ and $\varepsilon_{t,q}$ is that the transition to positive product innovation effort along the growth path does not happen all at once, but smoothly.

Evolution of Process and Product Innovation along the Growth Path

This last subsection describes how the share of R&D effort devoted to process and to product innovation evolves along the growth path of an economy with positive growth. In order to focus on the long-run behavior of the economy, consider some a generic period $t = t'$, which takes place very many periods after $t = 1$. Using Result 2, for this economy, in $t = t' - 1$, the equilibrium must have featured. $\lambda_{t-1}^* \in (0, 1)$. In addition, the level of λ_t^* will stem from the following FOC:

$$\frac{q_{t-1} + \rho\,(1 - \lambda_t^*)\,\sigma}{1 + \sigma A_{t-1} + \lambda_t^* \sigma}\frac{\sigma}{\rho} = \ln\left(1 + \sigma A_{t-1} + \lambda_t^* \sigma\right). \tag{62}$$

Unlike the benchmark model, when the equilibrium in t can actually feature *both* process and product innovation effort at the same time, keeping track of the values of q_{t-1} and A_{t-1} implies following the whole sequence of equilibria via the expressions in (56). The exact solution of λ_t^* in any given period t requires then solving (55) for each of the t, starting from $t = 1$, and keeping track of how the values of q_{t-1} and A_{t-1} evolve accordingly. In particular, it is

[22] When this fails to happen, the model will behave exactly as that in the main text, with process innovation effort taking place during a finite number of periods, and growth eventually coming to a halt in finite time.

quite straightforward to observe from (62) that $\partial\lambda_t^*/\partial A_{t-1} < 0$ while $\partial\lambda_t^*/\partial q_{t-1} > 0$. However, when $\lambda_{t-1}^* \in (0,1)$, both A_{t-1} and q_{t-1} will be growing at the same time, and thus the dynamic behavior of λ_t^* turns into a relatively complex quantitative exercise. We can nonetheless describe the long-run tendency of λ_t^*, in a similar fashion as done for the benchmark model in Proposition 5. The result below shows that, even when firms can undertake both process and product innovation effort at the same time, the model would still lead to long-run dynamics where the importance of product innovation effort tends to gradually increase over the growth path.

Result 3 *Consider an economy that is able to sustain positive R&D effort in the long run. The share of R&D effort devoted to product innovation tends to grow relative to its historical average when considering sufficiently long time horizons. More precisely, let* $\mathcal{H}_T=\{1,2,...,T\}$ *denote the set of periods starting in* $t=1$ *and ending in* $t=T$*, and suppose that* T *is a large number. Then, there exists some* $\widetilde{\Delta} \geq 0$*, such that:* $\frac{1}{T+\Delta}\sum_0^{T+\Delta}\lambda_t^* < \frac{1}{T-1}\sum_0^{T-1}\lambda_t^*$*, for* $\Delta \geq \widetilde{\Delta}$*.*

Proof. *Consider a finite time horizon* T*, a suppose* T *is a very large number. Let* $\Gamma_{T-1} \equiv \frac{1}{(T-1)}\sum_0^{T-1}\lambda_t^*$*. Given that* T *is a very large number, it follows that*

$$q_{T-1} \simeq \rho(T-1)\left(1-\Gamma_{T-1}\right), \tag{63}$$

$$A_{t-1} \simeq (T-1)\Gamma_{T-1}. \tag{64}$$

Replacing (63) *and* (56) *into* (62) *applied to* T*, we can obtain as FOC for* λ_T^* *the following expression*

$$\frac{T\left(1-\Gamma_{T-1}\right)+\Gamma_{T-1}-\lambda_T^*}{1+\sigma(T-1)\Gamma_{T-1}+\lambda_T^*\sigma}\sigma = \ln\left[1+\sigma(T-1)\Gamma_{T-1}+\lambda_T^*\sigma\right]. \tag{65}$$

Suppose now that $\lambda_T^* = \Gamma_{T-1}$*. Replacing this into (65) leads to:*

$$\frac{\sigma\left(1-\Gamma_{T-1}\right)T}{1+\sigma\Gamma_{T-1}T} = \ln\left(1+\sigma\Gamma_{T-1}T\right). \tag{66}$$

Now, consider the set of periods $\{T,...,T+\Delta\}$*, and suppose that* $\lambda_t^* = \Gamma_{T-1}$ *during all* $t \in \{T,...,T+\Delta\}$*. By a similar reasoning as* (66)*, this would require that*

$$\frac{\sigma\left(1-\Gamma_{T-1}\right)(T+\delta)}{1+\sigma\Gamma_{T-1}(T+\delta)} = \ln\left[1+\sigma\Gamma_{T-1}(T+\delta)\right] \tag{67}$$

holds true for all $1 \leq \delta \leq \Delta$*. However, notice that if* (66) *holds true in* T*, then* (67) *will fail to hold for values of* δ *that are large enough. In particular, the RHS of* (67) *will grow larger than the LHS of* (67) *when* δ *is large enough. As a result, for values of* δ *that are large enough, it follows from* (62)*, that we will observe* $\lambda_{T+\delta}^* < \Gamma_{T-1}$*, which in turn implies that* $\frac{1}{T+\Delta}\sum_0^{T+\Delta}\lambda_t^* < \frac{1}{T-1}\sum_0^{T-1}\lambda_t^*$*, for values of* Δ *sufficiently large.* ■

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